CHAPTER I

DIGITAL ARITHMETIC

¶ 1. The ordinary facts of elementary arithmetic are well known. It will be no trouble to most people who read this book to say that

$$9+7 = 16,$$

$$28+35 = 63,$$

$$9\times 7 = 63,$$

or even

$$17\times 12 = 204.$$

The rules are familiar and 'counting' is, essentially, easy.

What we propose to do in this chapter is to accept this ordinary arithmetic, but to modify it in one way. The result of this modification will be to make *computation* easier but, in return, to make *thought* harder for the beginner because of the element of abstraction that now comes in.

The new process will be called *digital arithmetic*. The reason for this name is that the operations of 'addition' and 'multiplication' with which we shall be concerned are to be the normal operations familiar in elementary arithmetic, *save that for the answers at each step only the units digit is to be retained.*

For example, whereas normally

$$8+7 = 15,$$

$$9+3 = 12,$$

we shall simply take

$$8+7 = 5,$$

$$9+3 = 2;$$

and whereas normally

$$8 \times 6 = 48,$$
$$7 \times 3 = 21,$$

we shall simply take

$$8 \times 6 = 8,$$
$$7 \times 3 = 1.$$

Note at once that we have radically altered the meanings of the two symbols $+$ and $\times$ and of the corresponding operations of addition and multiplication. In order to give a meaning to, say,

$$7+9,$$

we do *not* imagine seven apples and nine apples put into a box which thus contains sixteen apples; the physical interpretation has disappeared. We have said instead that $7+9$ is to mean 6 (a number reached by the process already described); and that is all about it. Our problem is not to justify the definition itself, for we are entitled to give any clearly-stated meaning that we wish to our words and symbols, but to justify instead the possibility of *using* the definition consistently and with significance. We therefore begin by examining the consistency of the addition.

¶ 2. Digital Addition

The laws of elementary arithmetic require us to add any collection of numbers without ambiguity. Suppose, for example, that we are given the three numbers

$$9,\ 7,\ 18.$$

We should say at once that the sum is 34 and should probably mean by that statement that

$$9+7+18 = 34.$$

On further reflection, we should probably agree also that the statement equally means any of the following:

$$9+18+7 = 34,$$
$$7+18+9 = 34,$$
$$7+9+18 = 34,$$
$$18+9+7 = 34,$$
$$18+7+9 = 34.$$

In other words, *the three numbers* 9, 7, 18, *taken in any order*, *always add up to the same number* 34. A corresponding result holds similarly for any other set of numbers.

DIGRESSION. It may be helpful to give now as a digression what is in fact the ultimately basic property of addition. The fundamental facts of elementary arithmetic (as popularly understood) are a whole collection of answers obtained successively by adding *two* numbers at a time. For instance, to add

$$3+7+11+19+5,$$

we should probably say something like this:

$$3+7 = 10,$$
$$10+11 = 21,$$
$$21+19 = 40,$$
$$40+5 = 45.$$

At any rate, the process would almost certainly involve the addition step by step of *two* numbers at any one time.

Reverting to the simpler case of three numbers only, say

$$3+5+9,$$

there thus appears a choice:

(i) to group $3+5$ and then add 9, giving

$$8+9,$$

or (ii) to take 3 and then add to it the group $5+9$, giving

$$3+14.$$

The point is, of course, that these two groupings give the same answer. In other words,

$$(3+5)+9 = 3+(5+9).$$

More generally, *if a, b, c are any three of the numbers of ordinary arithmetic, and if the symbol + has its ordinary meaning, then*

$$(a+b)+c = a+(b+c).$$

This law, governing the two ways in which the three numbers can be associated for addition two at a time (without changing order) is called the *associative law* for addition.

It will have been noticed that not only can the numbers be grouped in pairs according to the associative law, but also, as the preceding statement of that law implied, that the actual order in which the numbers are written is also immaterial. For example,

$$5+9 = 9+5;$$

and, more generally, for ordinary arithmetic

$$a+b = b+a.$$

Two numbers whose order can be changed in this way are said to *commute* and the general law

$$a+b = b+a$$

is called the *commutative law* for addition.

EXAMPLE

Check mentally that the two laws just enunciated ensure that the sum of the five numbers

3, 7, 12, 15, 19

is independent of the order in which they are selected.

Returning to *digital* addition, consider a sum such as

$$5+7+9+3+8.$$

The normal sum is 32 and so the digital sum is 2. The problem, however, is: do the two laws, the associative and the commutative, still hold when the arithmetic is digital? The answer is clearly *yes*; for what is true of the complete numbers must *a fortiori* be true of the units digits. Hence *it remains true in digital arithmetic that the associative law for addition*

$$(a+b)+c = a+(b+c)$$

and the commutative law for addition

$$a+b = b+a$$

retain their validity.

Consequently *all arithmetical manipulations dependent on them, and valid in ordinary arithmetic, are equally valid in digital arithmetic.*

EXAMPLE

Check mentally that these two laws ensure that the 'sum' of the numbers

2, 5, 8, 3, 7

is independent of the order in which they are selected.

¶ 3. Digital Multiplication

The corresponding treatment of digital multiplication may be discussed more briefly. The analogous laws are:

the *associative law* for multiplication,

$$(a\times b)\times c = a\times(b\times c)$$

or, more briefly,

$$(ab)c = a(bc);$$

and the *commutative law* for multiplication,

$$a\times b = b\times a,$$

or

$$ab = ba.$$

Since these laws hold for ordinary multiplication, they necessarily hold also for the units digits in the products.

EXAMPLE

Check mentally that these two laws ensure that the 'product' of the numbers

$$3, 5, 8, 4, 2, 6$$

is independent of the order in which they are selected.

¶ 4. DIVISORS OF ZERO

Without over-emphasising the point for the present, it is of interest to demonstrate at once a feature wherein digital arithmetic differs radically from ordinary arithmetic.

In ordinary arithmetic, as is well known, a relation

$$ab = 0$$

cannot hold unless at least one of a, b is zero. This is, for example, the basis of the argument used to finish the solution of a quadratic equation once it has been reduced to the form, say,

$$(x-1)(x-2) = 0.$$

The ending is 'Either $x-1 = 0$ or $x-2 = 0$, and so the equation is solved when $x = 1$ and when $x = 2$.

Note that this argument is only possible when the right-hand side is zero. It is *not true* that the equation

$$(x-1)(x-2) = 3$$

can be completed by the argument 'Either $x-1 = 3$ or $x-2 = 3$, and so the equation is solved when $x = 4$ and when $x = 5$.'

EXAMPLES

1. Prove that, if c is not zero and if a, b are different, then there is no set of numbers a, b, c for which the equation

$$(x-a)(x-b) = c$$

can be completed by the argument: 'Either $x-a = c$ or $x-b = c$ and so the equation is solved when $x = a+c$ and when $x = b+c$.'

2. Prove that, on the other hand, the equation

$$(x-3)(x-4) = 2$$

is satisfied (as to *one* solution) by setting $x-3 = 2$, so that $x = 5$; and that the equation

$$(4-x)(x-1) = 2$$

is satisfied (as to *both* solutions) by setting $4-x = 2$ or $x-1 = 2$, so that $x = 2$ or 3.

Examine the way in which these are specially constructed 'freak' equations and invent similar abnormalities yourself.

Returning to the main problem, we register the fact that, in ordinary arithmetic and algebra, a product cannot be zero unless one of its factors is. With digital arithmetic, however, the case is very different, in virtue of the four products

$$5\times 2 = 0, \quad 5\times 4 = 0, \quad 5\times 6 = 0, \quad 5\times 8 = 0.$$

In other words, *in digital arithmetic it is possible for the product*

$$ab$$

to be zero although neither factor is. This can happen when either one of a, b is 5 while the other is an even integer.

DEFINITION. A number a is called a *divisor of zero* when another number b exists such that the product ab is zero although neither individual factor is.

An immediate, and startling, consequence of the existence of divisors of zero is the possibility of producing in digital arithmetic *a quadratic equation with four distinct roots*:

Take, for example, the equation

$$(x-1)(x-2) = 0.$$

It is, of course, satisfied as usual by the two solutions

$$x = 1, \quad x = 2.$$

But there *may* also be a solution given by

$$x-1 = 5$$

or by

$$x-2 = 5,$$

provided that the second factor is an even integer. When $x = 6$, the left-hand side is 5×4, or zero; when $x = 7$, the left-hand side is 6×5, or zero. Hence *the quadratic* equation

$$x^2-3x+2 = 0,$$

or

$$(x-1)(x-2) = 0,$$

has the *four* roots 1, 2, 6, 7.

EXAMPLES

1. Find all the solutions of the equations

$$\text{(i)}\ x^2-4x+3 = 0,$$

$$\text{(ii)}\ x^2-5x+4 = 0,$$

$$\text{(iii)}\ x^2-5x+6 = 0,$$

$$\text{(iv)}\ x^2-6x+8 = 0.$$

2. Prove that the quadratic equation

$$5x^2+5x = 0$$

is satisfied by all integral values of x. Explain why it is not permissible to divide throughout by 5 without further examination.

5. The Distributive Law

In order to demonstrate quickly the unexpected properties of digital arithmetic we have, in fact, glossed over one or two theoretical points of detail that now require attention. The first of these is the *distributive law*

$$\begin{cases} a(b+c) = ab+ac, \\ (b+c)a = ba+ca, \end{cases}$$

which enables us to 'remove brackets'. The truth of the law follows, as in the previous cases, from the fact that the law, being true for ordinary arithmetic, must, in particular, be true for the units digits.

It was this law that enabled us to use a sequence of argument (now given in detail) such as

$$\begin{aligned} &(x+1)(x+2) \\ &= (x+1)(x)+(x+1)(2) && \text{distributive} \\ &= x^2+x+x(2)+1.2 && \text{distributive} \\ &= x^2+x+2x+2 && \text{commutative} \\ &= x^2+(x+2x)+2 && \text{associative} \\ &= x^2+3x+2. \end{aligned}$$

From now on we shall normally use the associative, commutative and distributive laws without comment. In other words, the manipulation will 'look like' ordinary algebra.

6. The Language of Sets

Before passing to the next point of detail, we ought perhaps to say a few words about sets. Not much is required at present, but the language is convenient and leads to precision.

The effect of the definition of digital arithmetic is to replace the infinite sequence of numbers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

by the *ten integers*

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

(The fate of the negative numbers is the next thing to be considered.) By ordinary use of language, we may say that the 'set'

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

has been replaced by the 'set'

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

These ideas give us two things—a *name* and a *notation*. The word *set* is used to denote any collection of objects whatever, subject to the sole requirement of a *rule* to determine whether a given object is a member or not; for example, *the set of all plane triangles* is defined: any given equilateral triangle is a member, but no circle can be.

A set may be designated for reference by any convenient letter, usually capital, and its members may be exhibited by enumeration within braces { }. Thus the set with which we are dealing in digital arithmetic may be denoted by the letter D, where

$$D \equiv \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$