CAMBRIDGE TRACTS IN MATHEMATICS

68. *Gibbs states on countable sets*

TO CECILIA

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Gibbs states on countable sets

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Contents

Preface

In the last few years there has been a great deal of interest in problems arising in classical lattice statistical mechanics. The aim of this book is to provide to mathematicians with no background in physics an introduction to some of the results in this field. As the average mathematician probably has difficulty in understanding the language of mathematical physics, the approach of the book is to consider the subject as a branch of probability theory. It is thus assumed that the reader is acquainted with some of the basic facts of probability theory (e.g. σ-algebras, probability measures, finite state Markov chains), but apart from this the material is self-contained.

The basic objects to be studied will be certain classes of probability measures on $\mathscr{P}(S)$, where S is a set (finite or countably infinite) and $\mathscr{P}(S)$ denotes the set of subsets of S. The points of S can be interpreted as sites, each of which can be either empty or occupied by a particle, and the subset of $A \in \mathscr{P}(S)$ can be regarded as denoting when there are particles at exactly the points in A. Thus the probability measures on $\mathscr{P}(S)$ describe the distribution of configurations of particles; and they will usually represent the equilibrium distribution of some physical model.

There are three parts to the book: the first part is Chapters 1, 2 and 3, and in this the points of S are the vertices of a finite graph; the second part is Chapter 4, where the points of S are the vertices of a countable graph; the rest of the book constitutes the third part, in which the set S has no additional structure.

The first two parts deal with models where the interaction of the particles has some connection with the graph structure, namely the interaction is only between particles occupying sites of the graph that are nearest neighbours. This leads to a class of measures on $\mathscr{P}(S)$ called Markov random fields. It is shown in the first two parts that this class is the same as another class of measures (which arise in statistical mechanics), the Gibbs states given by nearest neighbour potentials.

The third part also deals with Gibbs states, in this case with the graph structure removed. In this part models are considered with the following property: if Λ is a finite subset of S, $A \subset \Lambda$ and $X \subset S - \Lambda$, then the conditional probability of there being particles on Λ at exactly the points of A, given that on $S - \Lambda$ there are particles at exactly the points of X, is specified. (This says that if we know what is happening outside a finite subset of S then we can compute the distribution of particles inside the finite set.) Denote the above conditional probability by $f^{\Lambda}(A, X)$. The relations which the $f^{\Lambda}(A, X)$ must satisfy are determined in Chapter 5, and in the following chapters an attempt is made to find out under what conditions the $f^{\Lambda}(A, X)$ do or do not uniquely determine a probability measure on $\mathscr{P}(S)$. This possible non-uniqueness corresponds in the language of statistical mechanics to the phenomenon of phase transition.

There is a slight duplication between the second and third parts in order to make the third part independent of the rest of the material (and thus it is possible to start reading the book at Chapter 5). At the end of most of the chapters there are some notes which refer to the Bibliography at the end of the book. The Bibliography is by no means complete but should serve as an introduction for someone new to the subject; for further new results the reader should look at the current issues of journals like *Communications in Mathematical Physics* and *The Journal of Mathematical Physics*.

This book grew out of some notes written in the summer of 1972. I am deeply indebted to Frank Spitzer who introduced me to statistical mechanics and taught me a lot of the ideas and techniques used in the book, and to Adriano Garsia for teaching me probability, and to both for advice and encouragement. Thanks also to John Kingman and Geoffery Grimmett for reading the manuscript and for their helpful comments. The work on the material in the book was done between July 1971 and January 1973, and the author acknowledges support from Air Force Grant AF-AFOSR-2088 at the University of California, San Diego, and also a Science Research Council Fellowship and an IBM Research Fellowship at the Mathematical Institute, Oxford University.

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