INTRODUCTION TO DYNAMICS

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BY

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PREFACE

On several occasions during recent years I have given in Cambridge a short course of lectures on elementary dynamics. In these lectures I have suffered some embarrassment from the lack of a suitable text-book to which the class could refer for further information when, from lack of time, I was unable to treat some particular topic as fully as I should have wished. There exist, indeed, many books of something like the right scope, but none that I know has what seems to me the right modern approach to the subject. This book is designed to fill the gap.

The lectures just mentioned were for first-year undergraduates reading for Part I of the Mathematical Tripos, and the arrangement of the lecture-list is such that the class had previously attended a course in elementary statics. This arrangement of the lecture-list is no doubt open to criticism, but it has great practical advantages, and in this book I shall follow the same plan, i.e. I shall assume that the reader already has some knowledge of elementary statics. In theory, no previous knowledge of dynamics is required, though I daresay that in practice some previous acquaintance with the bare elements of the subject might be desirable. The only other equipment required of the reader is a working knowledge of the elements of the calculus.

The scope of the book is fairly clearly defined as the study of motion in two dimensions—particle, rigid body, system—without Lagrange's equations. It is intended as a general introduction to the subject, and I have made no attempt to adhere to the syllabus for any particular examination. I hope that the treatment is such that the reader can proceed naturally and uninterruptedly to the more advanced parts of the subject, that there is nothing he will wish to forget and nothing he will have to unlearn.

There are still some elementary books that seem to suffer from a hangover from the old days of 'mechanics without the calculus'. They use the methods of the calculus as little as may be, and when they do use them it is half-apologetically. I suggest that this attitude is dangerously out-of-date. Nowadays all students starting on the serious study of mathematics at the University have already a pretty good knowledge of the calculus and of mathematical analysis, and an opportunity to use these techniques as a tool for solving the problems of natural philosophy is invaluable. It assures the student (if he needs assurance) of the importance and usefulness of the calculus, and it gives him valuable practice in its applications. I would push the argument even further, and suggest that it is far better to over-estimate the reader's knowledge of analysis than to under-estimate it. If he occasionally meets a theorem he has not seen before, so much the better; he will be all the more readily convinced of its usefulness and importance.

The book is essentially elementary, and this explains the choice of the way in which the subject is developed. One question that arises continually is about 'the general and the particular'—about particular applications of general theories. Is it better to explain the general theory first and pick out the particular case afterwards for special study? Or is it better to treat the special case first as an introduction to the general theory? There is clearly no plain and unequivocal answer to this question; it depends on the topic with which we are dealing and on the audience for whom the exposition is designed. In this book, which is intended to be elementary, I have usually treated the special case before the general theory. This is because I have aimed at finding the most easily comprehended exposition rather than the most concise. Thus (setting aside for a moment the three introductory chapters, which are somewhat apart from the rest of the book) I deal with the motion of a particle on a straight line before motion in a plane. I put the problem of constrained motion in a plane before the problem of free motion. I consider the motion of a rigid body before the motion of a general dynamical system. Even in the motion of a single rigid body I particularize still further, and deal with the special problem of rotation about a fixed axis before the general problem of motion in a plane.

Experience confirms, I think, that this approach is congenial to the students in the early stages of the subject. It does, however, involve a certain amount of repetition. For example, the equation of energy appears several times—for a particle moving on a line, for two particles on a line, for a particle in a plane, for a rigid body, for a system. But this repetitiveness is not altogether a disadvantage. The theorem first appears in a very simple form, and the more recondite cases present themselves as natural generalizations of results already familiar; its reappearance in different contexts serves to impress on the reader its universality and importance.

The vexed question of vectors, or rather of vector notation, has lately been much discussed, and the disputants have been seen maintaining with no little heat their various opinions. I am aware that the *via media* I have chosen in this book will be uncongenial to the extremists of both parties. One party stands for vector notation all the time—even for problems to which it is not particularly well adapted. In some expositions the whole emphasis seems to be on the manipulation of vector formulae rather than on dynamical principles, and we occasionally meet a proof that looks like a stunt or a conjuring trick. This party seems to regard the invention of Cartesian axes as a regrettable accident, a slightly indelicate matter seldom alluded to in polite society. The other party takes the diametrically opposite view. Its members seem always ill at ease with the vectors (just as writers a century ago were ill at ease with the complex variable) and they breathe a sigh of relief when they can get back—which they do as quickly as possible—to the friendly shelter of x, y, z. In this book I have tried to avoid both these extremes. What is of primary importance is a thorough understanding of the dynamical principles, and I have taken what seems to me the common-sense line—that is to say, I have used vector notation freely when it substantially shortens or clarifies the argument. But I have not insisted on it everywhere, because there will be some readers to whom it is unfamiliar, and these readers will make quicker progress if the unfamiliar notation is not used too freely. As I have said, I expect of the reader a fair knowledge of the calculus, which is essential to a proper understanding of the subject. A wide knowledge of the geometrical vector theory is not of comparable importance at this stage; it comes into its own in the field theories of electricity and hydrodynamics. In particular, I do not often use the notation of the vector product. I use the scalar product freely, but in two-dimensional work we can, if we wish, dispense with the notion of the vector product. The reason is, in essence, that the vector product of two vectors lying in a given plane is perpendicular to that plane, and a set of such vectors, all perpendicular to the plane, can be treated effectively as scalars. Thus, for example, I usually write the moment about O of a force (X, Y) at (x, y) as a scalar in the form $xY - yX$. The reader who is expert in the geometrical vector theory is at liberty to write it in the usual form of a vector product if he prefers to do so.

I have included a careful study of the differential equation

$$\tfrac{1}{2}\dot{x}^2 = f(x)$$

which occurs so frequently in dynamics. It is vitally important to be able to infer the general nature of the solution from a glance at the graph of $f(x)$. Then, when the general nature of the solution has been understood (but not sooner) we shall usually wish to determine the explicit relation between x and t. We can attempt to do this by separation of the variables; but I have emphasized that for this equation, and for some others as well, it is often expedient to replace the dependent variable x by a new dependent variable suitably chosen.

A word about the general structure of the book, and about some of its special features. To begin with there are three introductory chapters. They are concerned with the basic notions required in dynamics—vector quantities, vector calculus, and the fundamental ideas of Newtonian dynamics. I expect that some readers will know the content of these chapters sufficiently well already, and for them the book effectively starts with Chapter IV. The first three chapters are to be regarded as an introduction to the subject as a whole, not merely to the present book, and for this reason the space with which we deal in these chapters is three-dimensional. In the following chapters we are mainly concerned with two dimensions—with the motion of a particle in a plane, or of a rigid body or system moving in such a way that each particle moves parallel to a given plane. We begin with particle dynamics.

At first sight it may appear that the balance is wrong—that too much attention is given to the rectilinear problem. But this appearance is, I think, illusory. The point is that many of the ideas that we meet in the special case reappear later. They are considered rather fully on their first appearance, so that later the results can be quoted *tout court* without further discussion. I discuss the theory of variable mass for the rectilinear case only; I confess that it seems to me that the importance of this topic has sometimes been over-estimated. I have given rather more space to the theory of orbits than has been usual in books of this scope, but I think this is justified, both by the intrinsic interest of the subject, and by the importance of it in the history of dynamics. In particular, the study of the Newtonian orbit has so permeated the whole subject that a fairly full account of it seems to be essential. I have eschewed the tiresome 'p, r equations' that still find a place in some elementary books.

I give a fairly full account of the motion of two particles in a plane—a simple example of a general dynamical system. The rather detailed discussion of the collision problem occupies more space than is usually devoted to this topic, but this is justified, I think, by the importance of the subject in mathematical physics.

I begin the study of rigid dynamics with a discussion of the motion of a rigid lamina in its plane, an important and elegant theory, packed with new and stimulating ideas. It has been on the whole rather neglected in English text-books.

Finally, I discuss the motion of a system, but without the general analytical theory of the Lagrange equations. Here we are faced with a fundamental difficulty—how far is it worth while to go with only the elementary techniques? I doubt if there is a final answer to this question. There are, of course, plenty of interesting systems, of a fairly simple type, that can be studied without further resources. But the methods are *ad hoc* methods, and the theory lacks the reassuring sense of completeness that we feel in the theory for a particle or in the theory for a single rigid body. The natural object to aim at would be to do as much as is interesting and useful without straining the available tools to do jobs for which they are fundamentally unsuited. I hope my judgment of this delicate matter is not too wide of the mark.

In the matter of notation I have been on the whole conservative. I do not venture to disturb such time-honoured forms as $\mathbf{P} = m\mathbf{f}$, or $T + V = C$. I have aimed at a reasonable standard of consistency of notation, but I would point out that insistence on consistency of notation can be overdone. The student should not become so much the slave of a particular convention that he feels completely at sea when he reads an author who uses a different convention. I have taken the potential V as fundamental in describing a conservative field of force rather than the work-function U, so I usually write $X = -\partial V/\partial x$ rather than $X = \partial U/\partial x$. The advantage of V is its physical significance, as in the equation of energy just mentioned. It has sometimes been maintained that it is expedient to write $X = -\partial V/\partial x$ for the electric field, the minus sign on the right being natural and proper, but that for the mechanical field the same minus sign is an intolerable nuisance. I have never been able to treat this view very seriously.

I have included in the text a number of worked examples to illustrate the application of the theory to concrete problems, and I have added short collections of examples for practice at the ends of the chapters. Most of these are taken from, or are based on examples taken from, Part I of the Mathematical Tripos, or from the examinations for Entrance Scholarships at Cambridge. I have to thank the Cambridge University Press for permission to reprint them here. Among the examples a few of a rather higher standard of difficulty are indicated by an asterisk. The reader should test his mastery of the theory by trying his hand on the examples at the ends of the chapters; but he should guard against the attitude of mind, encouraged by text-books of a certain stamp, that regards the study of dynamics (or of any other branch of mathematics) as preparation for an examination, and puts as the first object the learning of examination tricks. This attitude of mind is not only bad in itself, but, strangely enough, it does not even pay. The candidate who relies on examination tricks is often defeated by a quite simple problem if it lies a little off the beaten track. The best preparation for the examination is a proper understanding of the theory. Here we have an example of the pleasing situation (so different from the affairs of everyday life) where the highest principles are also the most profitable.

I have taken particular care to give a proper distribution of emphasis to the various parts of the subject, and I hope I have given a just impression of the relative importance of the various topics. I have put plainness before elegance, and have occasionally admitted infelicities of style in the interests of sheer clarity. I hope that the reader who has mastered this book will be in a good position to pursue the subject into its higher developments—a pilgrimage that will bring him an abounding reward.

A large number of friends and pupils have helped me with various details at different stages, and it is impossible to mention them all here. I am deeply conscious that the book would have been much inferior without their help. In particular, I would express my thanks to Dr J. Bronowski, Mr P. Hall, Mr A. E. Ingham, Dr D. R. Taunt, and Mr A. J. Weir. For help with the diagrams I am indebted to many friends, especially Mr J. H. Halton and Mr G. Matthews. Finally, my best thanks to Dr Taunt and to Dr W. B. Pennington for their assistance in the onerous task of proof-reading.

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21 *May* 1951

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