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INTRODUCTION

Sunt utique viri magni, de quibus non est temere pronuntiandum.
Wallis to Collins, 16 September 1676

Leibniz came only relatively late to any real acquaintance with the central mathematical notions of his day. By their special nature his early studies had merely awakened in him an interest in the deductive method of mathematics and the formal aspects of computation, while his technical knowledge long remained limited and superficial. It was neither in adolescence – like Pascal and Huygens – nor, like Torricelli and Newton, in his student days at university, nor even – as John Wallis before him – in his first graduate years that he entered the mathematical area, but rather in full intellectual maturity, his doctorate gained and with a developed awareness of his abilities and creative potentialities. The very reverse of the usual French image of a Germanic scholar as a heavy moralizing pedant, he rapidly became on his arrival in Paris in 1672 the charming, irresistibly fascinating centre of everyone's admiring attention, sparkling in his conversation and brilliant in his quick wit – at once an accomplished diplomat subtly skilful in promoting his ambitious political aims, and a vivacious young man of the world with an engaging zest for life and an infinite capacity for hard work. He joins in disputation with the Cartesians one day, with the Jansenists or Jesuits the next, dispatches in a few hastily written words a difficult diplomatic commission, shows interest in the latest inventions, is the guest of craftsmen, magicians, scholars, courtiers and charlatans, writes sharply detailed political reports, dreams of reforming the law on a natural basis, even of radically transforming the existing order of society of his day – and in every such activity he is totally absorbed, knows everything, grasps it all in his mind, has a thousand threads in his hands. It is to be his fate never to find firm ground under his feet, never to be able to create freely but always to remain a tool in the hands of men in power who know how to use him for their purpose but who are quick to set insurmountable barriers to his activities when he threatens to become troublesome.

A mind of such riches, such depths of knowledge, ability and experience, impressed with the necessity of gathering, comprehending and unifying needs a system, a *filum cogitandi* – an intellectual framework into which everything he thinks or encounters can be fitted. How could he be satisfied with the rigid, purely classificatory method of the late scholastics which he had mastered well enough at the German universities? Not even the system of the sciences promulgated by Descartes and then influencing all seriously enquiring minds of the day can altogether please him; he looks further and he looks deeper. In his opinion great progress has been made by Galileo and Descartes, both taking their start in nature, both unveiling her secrets in new ways and by new means. But did they penetrate to the deepest reaches? And is not the mechanistic basis on which they built their world concept too narrow? Will not Cartesianism become calcified just as the now slowly retreating Aristotelianism had done? Leibniz wants to create not a rigid edifice but an elastic structure which shall stand on firmer ground and be more durably constructed. To accomplish this purpose our mode of reasoning must itself be simplified, a method must be found which can comprehend and typify the essential features of the thought process. To think clearly and correctly must no longer be the prerogative of a few chosen spirits but must become the common property of all educated men. And Leibniz believes he is called to lead this march forward like a new Prometheus – struggling in the cause of his great art, the *ars inveniendi*, and ready to yield it up for the benefit of all mankind.

A man who places such thoughts into the forefront of his mind has mathematics in his blood even if he is still ignorant of its detail. And indeed, the actual breadth of knowledge of this twenty-six-year-old is still deplorable. With geometry he evidently gained no real acquaintance either at school or at university. In later years he often recalls how little knowledge in this field he then acquired. No doubt the first book of Euclid's *Elements* (probably in one of the widely used shortened versions by Clavius) was discussed in his presence, but his lessons in it made no impression on him.[1] From his school days

[1] According to Guhrauer (1846) I: 26 Leibniz heard lectures on Euclid by Kühn probably at the start of his studies in the faculty of philosophy, that is in the early summer of 1661; but because of their utter obscurity Leibniz alone, we are told, was somehow able to follow their course. In his memorandum 'De constructione' (winter 1674–5) Leibniz records that he was now reading attentively in Euclid's *Elements* having rarely done so before (*LMG* VII: 254). In his 'Historia et origo

Leibniz can remember only the insignificant *Arithmetic* by Lanz and the good, but limited, *Arithmetic* by Clavius.[2] As an undergraduate the two volumes of the well-commented Latin edition of Descartes' *Géométrie* by Schooten seemed much too complicated to him.[3]

In the *Disputatio arithmetica de complexionibus*[4] which he delivered as a new Magister artium in March 1666 at Leipzig university and which appeared still in the same year in print in a much enlarged version as the *Dissertatio de arte combinatoria*,[5] he mentions, apart from Descartes' *Géométrie*,[6] also Schooten's *Principia matheseos universalis*[7] and in addition (because its algebraic notation differs from his own) Barrow's edition of Euclid.[8] In technical detailed questions on combinations Leibniz starts from Schwenter–Harsdörffer's *Erquickstunden*;[9] from it he borrows references to Cardan,[10] Butéon's *Logistica*, Tartaglia's *General trattato* and Clavius' *Sphæra Johannis de Sacrobosco*,[11] but he has obviously read more closely only Cardan's *Practica arithmeticæ* where, as he emphasizes, the details mentioned by Schwenter do not occur:[12] that all given there is actually all but a literal translation from Clavius (though recognizable as such only to the expert) is briefly indicated. In Clavius' book[13] Leibniz had found remarks on combinations of letters. The title-page[14] itself of the *Dissertatio* has its origin in the illustration at the beginning of Clavius' first chapter. Of more recent foreign literature – Hérigone, Tacquet, Pascal – Leibniz is ignorant. True, he has at this period looked also at certain specialist mathematical books, but for the most part only turned their leaves without real attention; for a

calculi differentialis' (1714) he remarks in retrospect that initially, much occupied with other studies, he had not given sufficient attention to Euclid (*LMG* v: 398). His essay 'In Euclidis πρῶτα' (*c.* 1696) shows that Leibniz adheres to the Latin word forms of Clavius' edition of the *Elements* (*LMG* v: 183–211).

2 Leibniz–Jakob Bernoulli, April 1703, draft postscript (*LMG* III: 72).

3 *Ibid.* 72.

4 Critical edition by Kabitz in *LSB* VI. 1: 170–75, 228–30.

5 *Ibid.* 165–228.

6 *Ibid.* 171.

7 *Ibid.* 171.

8 *Ibid.* 173.

9 *Ibid.* 169, 173, 202–4, 215–18.

10 *Ibid.* 173, 178, 229.

11 *Ibid.* 173.

12 The passage does in fact not occur in the *Practica arithmeticæ* but in the *Opus novum*; see also *Opera* (1663) IV: 558.

13 Ch. 1: 'De numero et ordine elementorum', quoted with page reference in *LSB* VI. 1: 215.

14 *Ibid.* 166. The blockmaker has replaced the Jesuits' IHS-emblem of the original by a rose and has made a few other changes of little significance in redrawing it. *See* Knobloch (1971).

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1	0	1	2	3	4	...	12
2	0	0	1	3	6	...	66
3	0	0	0	1	4	...	220
4	0	0	0	0	1	...	495
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12	0	0	0	0	0		1
*	0	1	3	7	15	...	4095
+	1	2	4	8	16	...	4096

prolonged study of a lengthy chain of deductive reasoning he lacked patience. In the content of his *Disputatio* Leibniz scarcely goes beyond his model Schwenter. He gives the combinations of n elements taken p at a time according to the table above[15] and knows the addition law by which it is built up.[16] The *Dissertatio de arte combinatoria* which grew out of the disputation further contains the multiplication law for $n!$ permutations of n different elements[17] and a related table (up to $n = 24$). It continues with the relation

$$2(n+1)!-n.n! = (n+1)!+n!$$

and the enumeration of certain permutations with repeated elements[18]–not in a tight methodically arranged form, but expounded by means of examples from the doctrine of syllogisms,[19] from the combination of letters into words,[20] of musical notes into melodies,[21] of long and short syllables into lines of verse[22] and the like. At the end of the disputation, in a passage omitted in the revised dissertation, appears the metaphysical corollary drawn from Cardan:[23] *Infinitum aliud alio maius est*. In his *Arithmetica infinitorum*, Leibniz adds, 'Seth Ward' (read: Wallis) is said to have adopted a different standpoint. When an unauthorized reprint of the *Ars combinatoria* appeared in 1690 Leibniz had a notice inserted in the *Acta Eruditorum*[24] pointing out

15 *LSB* VI. 1: 174.
16 *Ibid.* 175.
17 *Ibid.* 211–12.
18 *Ibid.* 212–13.
19 *Ibid.* 180–8, in the course of a critical discussion of J. Hospinianus' writings on Logic of 1560 and 1576.
20 *Ibid.* 223.
21 *Ibid.* 218. An error due to carelessness in this calculation has been pointed out by M. Cantor in his *Vorlesungen über Geschichte der Mathematik* $_{2}$III: 44.
22 *LSB* VI. 1: 225–7.
23 *Ibid.* 229. Leibniz refers the reader to *Practica arithmeticæ*, ch. 66, Nos. 165 and 260.
24 *AE* (February 1691): 63–4 = *LPG* IV: 103–4.

the unfinished and juvenile character of the work in its treatment of many details – probably referring to his indiscriminate use of equality signs and similar faults.

Leibniz' application for his doctor's degree in law in Leipzig was refused in 1666 on account of his youth. Accordingly he went to Altdorf where his promotion[25] took place in February 1667. He subsequently stayed on in Nuremberg for several months, where for a while he moved entirely within a circle of Rosicrucians, trying his hardest to find out their secrets. Leibniz remembers two mathematical works he had in his hands during this time at Nuremberg. One is Léotaud's *Examen circuli quadraturæ*, the other Cavalieri's *Geometria indivisibilibus continuorum nova quadam ratione promota.*[26] For serious study there was, of course, no opportunity even yet. The second part of Léotaud's book, in which the erroneous circle quadrature of Grégoire de Saint-Vincent is refuted, he could in any case not have understood without a proper knowledge of the *Opus geometricum.*[27] The matter is different in regard to the first part, which contains a youthful piece (written a generation previously) by Artus de Lionne. In this, the *Amœnior curvilineorum contemplatio,*[28] curvilinear areas whose boundaries are circular arcs are squared in an ingenious way. The starting point is the quadrable circular lunule of Hippocrates cut off between the semicircle ACB and the quadrant ADB (fig. 1). A first major result is the equality of the mixtilinear triangle ACD and the rectilinear triangle AFN; a second result the equality of the mixtilinear triangles ACD and AED, each of which equals half the isosceles rightangled triangle ACE (fig. 2); a third result is the equality of the two lunules cut off between the semicircular arcs ACB, BFC, CGA to the rightangled triangle ABC whose sides are the diameters of the semicircles (fig. 3). Leibniz could remember none of these details in later life, but memory of the book itself lingered.[29]

[25] The dissertation *De casibus perplexis in jure* (Nuremberg 1666) which Leibniz probably had already completed in Leipzig provided the basis for the disputation of 15 Nov. 1666; the degree ceremony took place at the university of Altdorf on 22 Feb. 1667 (Müller–Krönert: 9). The origin of the titlepage which has doubtless been adapted from another work (*LSB* VI. 1: 233) has not yet been established.

[26] Leibniz–Jakob Bernoulli, 1703, postscript (*LMG* III: 72); 'Hist. et origo' (*LMG* V: 398); Leibniz–Conti, 9 Apr. 1716 (*LBG*: 278).

[27] Concerning this work which Leibniz mentions already in his edition of Nizolius (1670) (*LSB* VI. 2: 432) and in a note of winter 1671–2 (*ibid.* 480), but which he apparently has not yet seen himself, *see* below ch. **2**: note 9.

[28] For the contents of this treatise *see* Hofmann (1938).

[29] Leibniz–Jakob Bernoulli, 1703, postscript (*LMG* III: 72); Leibniz–Bodenhausen, 5 Oct. 1692 (*LMG* VII: 375); 'Hist. et origo' (*LMG* V: 398).

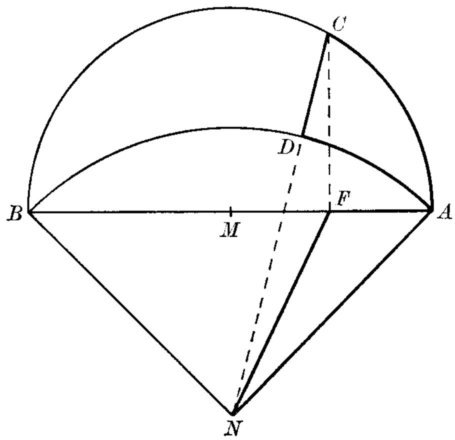

Fig. 1

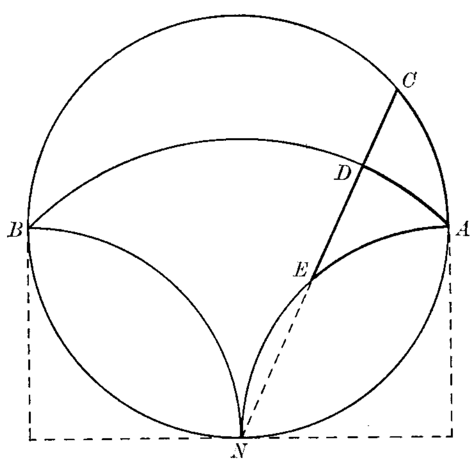

Fig. 2

In his 'proud ignorance' he believed himself capable of assimilating the contents of this and other mathematical books which he read 'like novels' by merely skimming through them.[30]

The ill effect of this skimping sort of study shows itself particularly in the case of Cavalieri's method of indivisibles which already contains within its compass a fair measure of vagueness and obscurity of its own. Leibniz did not at this time look too closely at the original

[30] Leibniz–Jakob Bernoulli, 1703, postscript (*LMG* III: 72).

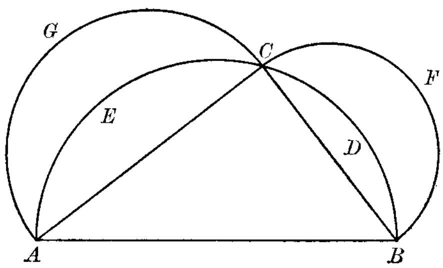

Fig. 3

but, we may add, effectively orientated his mathematical opinions in line with Hobbes' *Elementa philosophiæ*,[31] which, though a significant philosophical achievement, was nonetheless written by a man lacking proper mathematical expertise. The clearest indication of the level of Leibniz' mathematical knowledge of these days may be found in his *Hypothesis physica nova*[32] of 1671, where the method of indivisibles is briefly described as a *fundamentum prædemonstrabile*. Here two fundamental viewpoints, namely Cavalieri's concept of indivisibles and Archimedes' of finite increments, are everywhere confused – a feature of Cavalieri's book as well, though not so blatantly evident.[33] In an attempt to demonstrate the existence of an 'extensionless, indivisible' line-segment Leibniz constructs, quite perfunctorily, a continued bisection of a line.[34] He sets himself in conscious contrast to Euclid's well known definition: *Punctum est*

[31] Leibniz by 1670–1 knew the separate Latin parts of Hobbes' *Elementa philosophiæ* (1655, 1658, 1642) as well as the complete edition including the *Leviathan* (1651), the *Examinatio* (1660) and *De principiis* (1666). He refers to Hobbes' writings for the first time in his marginal notes (1663–4?) to Daniel Stahl's *Compendium* (1655) and Jakob Thomasius' *Philosophia practica* (1661) (*LSB* VI. 1: 22, 25, 60, 67). Several remarks in his letters of this time show that he was fascinated by many of Hobbes' ideas and greatly admired the subtleness of his thought. Hobbes' intention to represent all syllogistic conclusions by a symbolic calculus (*De corpore* I ch. 1, §2) touches closely on his own fundamental related ideas as does the expectation to find proofs for Euclid's axioms (*ibid.* ch. 6, §13); *see* below ch. **2**: note 12. Leibniz however strongly opposes the suggestion that definitions might be arbitrarily fixed: *see* p. 21 and Leibniz–Tschirnhaus, early 1680 (*LBG*: 405, 411). Regarding the relations between Leibniz and Hobbes *see* Couturat (1901): 457–72.

[32] Part 1: *Theoria motus concreti*, part 2: *Theoria motus abstracti*; *see* also ch. **3**: note 8.

[33] Without actually naming Cavalieri, Leibniz refers to the method of indivisibles in the preface to the *Theoria motus abstracti* (*LSB* VI. 2: 262). He had probably learnt of this method through his reading of Hobbes' *De corpore* I ch. 20, and at first appears to have informed himself about Cavalieri's procedure only at second hand – as he presumably also did with regard to Archimedes' writings.

[34] *LSB* VI. 2: 264.

cuius pars est nulla, and states that even though a point were an entity without an extension assignable in magnitude, or at all events less than any given quantity, it would still be possible to divide it; the parts indeed will be characterized as being without distance.[35] Other infinitesimal quantities can be regarded in a similar manner, for instance the indivisible of a circular arc.[36] This is certainly greater than its chord; while to a larger circle there pertains a larger arc-indivisible (on making the silent assumption of equal angle-elements in the two cases). In this way, Leibniz says, the puzzles of both the incommensurable and the angle of contact can be solved, particularly since every angle is an extensionless quantity.[37] The (regular) polygon of infinitely many sides indeed coincides, he says, with the corresponding circle, but only in extension and not in quantity, though the difference is less than any assignable quantity.[38] For an explanation of the measure of cylinder, cone and sphere the concept of movement is introduced;[39] but side-by-side with this idea of continuous quantities goes that of the discrete, as in the case of a point-by-point construction for the quadratrix or in Archimedes' measurement of the circle by polygons, where, as Leibniz says, it matters only to make the error immeasurably small.[40]

Leibniz is, as we see, completely under the spell of the concept of indivisibles, has no clear idea of the real nature of infinitesimal mathematics, and believes that for him, superficial conceptual allusions are sufficient. He further recounts that he had contrived during these early years a geometrical 'calculus' of his own, in which he operated with an unlimited number of squares and cubes,[41] unaware that it had all been done to far better purpose already by

[35] *Ibid.* 265 and 267. Here Leibniz refers to what he has read in Hobbes' *De corpore* (I ch. 15, §2), and *De principiis* (ch. 1). He sees in this new definition of a point the essential feature of Cavalieri's method.

[36] *Ibid.* 267.

[37] *Ibid.* 267; *see* also ch. **2**: note 9.

[38] *Ibid.* 267.

[39] *Ibid.* 272.

[40] *Ibid.* 273. Leibniz is rather proud at having successfully worked out the essentials of Cavalieri's concept. He writes to this effect on 1 (11) Mar. 1671 to Oldenburg, early in May to Velthuysen and again on 22 June (?) and 17 Aug. of the same year to Carcavy (*LSB* II. 1: 90, 97, 126, 143).

[41] Leibniz–Jakob Bernoulli, 1703, postscript (*LMG* III: 72). Leibniz there uses the technical term *quadratillum*, to be found also in his *Theoria motus abstracti* (*LSB* VI. 2: 275). These earliest mathematical studies – of which no written notes have survived – are mentioned again in connection with Hobbes' rather absurd objections to the validity of the so-called Pythagorean theorem (*De corpore* I ch. 20, prop. 1; *Examinatio*, dial. 3; *De Principiis* ch. 23); *see* *LSB* VI. 2: 267.

Viète and Descartes.[42] But as an offshoot of these first trials there developed a certain inward propensity towards mathematical lines of reasoning. With the more primitive things – such as a typical proof in elementary geometry or a lengthy transformation in algebra – he never even in later years found it easy to cope, and errors in calculations are no rarity in his writings. In his case we see with all clarity that purely formal schooling can only be accomplished during the years of youth. His failing in this regard, however, obliged him ever to think of new expedients and so to enrich science by new methods in situations where a capable average brain would have achieved his end without trouble. He was well aware of this and knew that his strength lay not in elaborating the formal side, but in finding the well-reasoned conclusion at the decisive point. Hence arises his striving for a mechanization of purely technical aspects, in particular the process of computation, hence his perennial struggle for a calculating machine equally useful for all four species of computation,[43] the idea of which – so difficult to translate into practice – was already in his mind before he came to Paris.

In the case of nearly all other important mathematicians the *grande passion* is already recognizable during puberty and leads in the period immediately following to decisive new ideas. In Leibniz' case this biologically significant period passed without any specifically mathematical experience; that came to him only in full maturity; but then indeed it took him by force, never to let him free again. During his four years in Paris he pursued mathematics as an autodidact with a strength and intensity known to few men. Chance has brought it about that this man, forgetful to an unprecedented degree in his individual mathematical results, carefully kept almost every sheet of his notes – many with the exact date, others datable through watermarks, the quality of the paper, the character of their handwriting, the notation employed (often changed in his early years) and technical terms used – a chaotic mass of scrap sheets from which Leibniz in his old age in all seriousness wanted to compile a description of the discoveries made or inaugurated by him.[44] Few of these notes have so

[42] Leibniz refers here to the progress made by Viète and Descartes in applying literal calculation to geometry.

[43] For details *see* von Mackensen (1969).

[44] Leibniz talks of a planned treatise with the title 'Scientia infiniti' in his letters to Johann Bernoulli of 31 Mar. 1694 (*LMG* III, 136); to Huygens of 22 June 1694 (*HO* **10**: 640); to L'Hospital of 16 Aug. 1694 (conjectured from *LMG* II: 252); and to Jakob Bernoulli of 12 Dec. 1695 (*LMG* III: 29). Johann Bernoulli mentions the planned work in the *AE* of October 1694: 393–8 (*BJC* I: 119), so does L'Hospital

far been published, many have to be regarded as preliminary studies and are unfit to be printed, but it has all only been very partially sifted or drawn upon. Still, it is now possible from the letters exchanged during those years to gain an insight, clear in all essentials, into Leibniz' mathematical development during his Paris period, and so into the birth of the higher analysis – and this must rank as one of the greatest products of human intellect. We should not overlook the difficulties obtaining at that time in publishing books; nor that learned journals were only then just beginning to appear and that editors preferred, on the one hand encyclopedic descriptions of the everyday world, and on the other, the cleverly styled essay. The letter in its immediacy of expression of personal experience or its outline indication of discovery, was therefore substitute for, and precursor of today's periodical. Several generations of scholars have centred their research effort on these letters, but only now is the full text of the originals coming to be known, only now has the history of their genesis been clarified by careful study of their detail. This by no means involves Leibniz alone but an interwoven network of mutual relations which it is necessary to understand and survey completely before the deeper significance of the whole can be comprehended.

In all innocence, Leibniz ultimately became involved in the disputes between French and English scholars which were concerned less with matters of science and learning than with personal touchiness and questions of national vanity. In his boundless optimism, Leibniz believed he could catch hold of the threads of this fine-spun net in one bold grasp, but he overestimated his own valour and misjudged the situation. Indeed from his ignorance of the true circumstances arose those ever regrettable misunderstandings which a generation later brought down upon him the severest of reproaches and which finally reached their climax in the accusation made against him of intellectual theft. That this was baseless – that he had in fact, like his great adversary Newton, reached his grandiose results by his own unaided ingenuity can now accurately be demonstrated through the documents, by the exact agreement of his

in the preface to his *Analyse* (1696): e, fols. 2r–2v. Leibniz writes on 16 Dec. 1697 to Bodenhausen, that the work might well be expected to be completed if only he had someone to assist him (*LMG* VII: 392). If not even a preliminary draft was put together, we may see the reason in the great number of related publications by Leibniz beginning in 1694, full of results which were leading him further and made a concentrated treatise dealing with fundamental concepts hardly feasible.