Cambridge University Press
978-0-521-08076-7 - Algebraic Topology-A Student's Guide
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Introduction

It is the object of this introduction to give a general survey of the material which faces the student of algebraic topology, and at the same time to give a guide to the sources from which this material can most conveniently be studied. It seems convenient to alternate between passages which comment on the material and passages which comment on the literature. When I have had to comment on a topic which has been treated by several authors, I have sometimes felt a responsibility to offer the student some guidance on which source to try first; I have done this by marking a recommended source with an asterisk. This does not mean that the other sources are not also good; some students may prefer them, and most will profit by seeing the same topic treated from more than one point of view. In some cases the marked source is chosen on the grounds that it gives a particularly short, simple or elementary account, while the others give longer, fuller or more advanced accounts.

In what follows, I shall refer to the following list of sources available in book form. A reference to the author's name, without further details, is a reference to this list.

J. F. Adams, 'Stable Homotopy Theory', J. Springer, 2nd ed. 1966 (Lecture Notes in Mathematics No. 3).

P. Alexandroff and H. Hopf, 'Topologie', J. Springer 1935.

M. André, 'Méthode Simpliciale en Algèbre Homologique et Algèbre Commutative', J. Springer 1967 (Lecture Notes in Mathematics No. 32).

M. F. Atiyah, 'K-Theory', Benjamin 1967.

A. Borel, 'Topics in the Homology Theory of Fibre Bundles', J. Springer 1967 (Lecture Notes in Mathematics No. 36).

H. Cartan and S. Eilenberg, 'Homological Algebra', Princeton University Press 1956 (Princeton Mathematical Series No. 19).

P. E. Conner and E. E. Floyd (1), 'Differentiable Periodic Maps', J. Springer 1964 (Ergebnisse series No. 33).

P. E. Conner and E. E. Floyd (2), 'The Relation of Cobordism to K-theories', J. Springer 1966 (Lecture notes in Mathematics No. 28).

A. Dold, 'Halbexakte Homotopie Funktoren', J. Springer 1966 (Lecture Notes in Mathematics No. 12).

J. Dugundji, 'Topology', Allyn and Bacon 1966.

B. Eckmann, 'Homotopy and Cohomology Theory', in Proceedings of the International Congress of Mathematicians 1962, Institut Mittag-Leffler 1963, pp 59-73.

S. Eilenberg and N. E. Steenrod, 'Foundations of Algebraic Topology', Princeton University Press 1952 (Princeton Mathematical Series No. 15).

P. Freyd, 'Abelian Categories', Harper and Row 1964.

P. Gabriel and M. Zisman, 'Calculus of Fractions and Homotopy Theory', J. Springer 1967 (Ergebnisse series No. 35).

R. Godement, 'Théorie des Faisceaux', Hermann 1958 (Actualités series 1252).

M. Greenberg, 'Lectures on Algebraic Topology', Benjamin 1967.

P. J. Hilton (1), 'An Introduction to Homotopy Theory', Cambridge University Press 1953 (Cambridge Tracts series No. 43).

P. J. Hilton (2), 'Homotopy Theory and Duality', Gordon and Breach, 1965.

P. J. Hilton and S. Wylie, 'Homology Theory', Cambridge University Press 1960.

F. Hirzebruch, 'Topological Methods in Algebraic Geometry (3rd ed., translated), J. Springer 1966.

J. G. Hocking and G. S. Young, 'Topology', Addison-Wesley 1961.

S. T. Hu, 'Homotopy Theory', Academic Press 1959.

W. Hurewicz and H. Wallman, 'Dimension Theory', Princeton University Press 1948 (Princeton Mathematical Series No. 4).

D. Husemoller, 'Fibre Bundles', McGraw-Hill 1966.

S. MacLane, 'Homology', J. Springer 1963 (Grundlehren series No. 114).

W. S. Massey, 'Algebraic Topology: An Introduction', Harcourt Brace and World, 1967.

J. P. May, 'Simplicial Objects in Algebraic Topology', Van Nostrand 1967 (Mathematical Studies series No. 11).

J. W. Milnor, 'Morse Theory', Princeton University Press 1963 (Annals of Mathematics Study No. 51).

B. Mitchell, 'Theory of Categories', Academic Press 1965.

R. S. Palais, 'Seminar on the Atiyah-Singer Index Theorem', Princeton University Press 1965 (Annals of Mathematics Study No. 57).

L. S. Pontryagin, 'Foundations of Combinatorial Topology', Graylock Press 1952.

H. Siefert and W. Threlfall, 'Lehrbuch der Topologie', Teubner 1934.

E. H. Spanier, 'Algebraic Topology', McGraw-Hill 1966.

N. E. Steenrod, 'The Topology of Fibre Bundles', Princeton University Press 1951 (Princeton Mathematical Series No. 14).

N. E. Steenrod and D. B. A. Epstein, 'Cohomology Operations', Princeton University Press 1962 (Annals of Mathematics Study No. 50).

R. G. Swan, 'The Theory of Sheaves', University of Chicago Press 1964).

E. Thomas, 'Seminar on Fibre Spaces', J. Springer 1966 (Lecture Notes in Mathematics No. 13).

H. Toda, 'Composition Methods in Homotopy Groups of Spheres', Princeton University Press 1962 (Annals of Mathematics Study No. 49).

A. H. Wallace, 'Algebraic Topology', Pergamon 1957.

G. W. Whitehead, 'Homotopy Theory', The M. I. T. Press 1966.

J. H. C. Whitehead, 'The Mathematical Works of J. H. C. Whitehead', Pergamon Press 1962.

In general, Spanier is the most useful single reference for the central core of the subject, followed by Husemoller for those topics which he treats.

1. A first course

I assume that most readers of this book will have had a first course in algebraic topology. This section, then, is included for completeness, and it can hardly escape a certain air of being directed at the teacher rather than the student. It is hoped that this slant diminishes in later sections.

A basic course in algebraic topology should certainly try to present a variety of phenomena typical of the subject. The author or lecturer should display a variety of spaces: cells, spheres, projective spaces, classical groups and their quotient spaces, function spaces .... Equally, one should display a variety of maps, that is, continuous functions between spaces. One must give the definition of homotopy, and one can then display a variety of phenomena or typical problems. First, we have classification problems, for example, the classification of maps $f:X \to Y$ into homotopy classes. (This can be illustrated by considering the case in which X and Y are the circle S^1; the existence and properties of the degree of a map $f:S^1 \to S^1$ can be stated as a theorem whose proof is deferred only a short time. One then has many applications to plane topology: the Brouwer fixed-point theorem for the disc E^2, the fundamental theorem of algebra, separation theorems, the topology needed for Cauchy's theorem in complex analysis, vector fields and critical-point theory in the plane.... But time presses one on.) Secondly, one has extension problems; the homotopy extension property comes in here, at least for simple pairs like the n-cell E^n and its boundary S^{n-1}. Thirdly, one has lifting problems; for this one must display and discuss fiberings, including coverings. (Some authorities prefer a separate preliminary discussion of coverings, probably in connection with the fundamental group; but personally I believe in going straight to fiberings, with coverings as an important special case.) One must also prove the homotopy lifting property, at least for simple spaces like the n-cube I^n. (At this point one can prove the theorem about the degree of a map $f:S^1 \to S^1$, by using the covering map from the real line R^1 to S^1.) By analogy with the word 'fibering', one introduces 'cofiberings' or 'cofibrations' in studying extension problems.

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The basic facts about homotopy are given in Dugundji chaps. 15, 18, Greenberg part 1, *Hilton (1) chap. 1, Hocking and Young chap. 4, Hu chap. 1, and Spanier chap. 1. For classification problems, see Hu chap. 1. For extension problems, see *Hu chap. 1 or Spanier chap. 1. For lifting problems, see *Hu chap. 1 or Spanier chap. 2. For fiberings, see Dugundji chap 20, *Hilton (1) chap. 5, Hu chap. 3 or Spanier chap. 2. For cofiberings, see Spanier chap. 1.

Of course one has to face the question, what is the good category of spaces in which to do homotopy theory? Personally, I believe that one should introduce CW-complexes into even a basic course; I would advocate going as far as the theorem that every map between CW-complexes is homotopic to a cellular map. Up to this point the material belongs almost wholly to analytic topology; this theorem is usually proved by simplicial approximation, but it can be proved by an ad hoc subdivision argument, subdividing the cube by hyperplanes parallel to its faces. (Such a subdivision has already been used to prove the homotopy lifting property.)

The material on CW-complexes may be found in Hilton (1) chap. 7, Spanier chap. 7 or G. W. Whitehead chap. 2. The best source, however, is probably the original paper by *J. H. C. Whitehead, and an appropriate extract is reprinted here (see Paper no. 1).

Next, one must certainly define absolute and relative homotopy groups, and prove some of their logically elementary properties (for example, the exact sequences of a pair and a fibering). Some authorities prefer a preliminary discussion of the fundamental group $\pi_1(X)$, but personally I believe in saving time here and defining the groups $\pi_n(X, A)$ for all n at one blow. Some authors might advocate proceeding in even greater generality, defining track groups, homotopy groups of maps and so forth; but if these are needed they can quickly be obtained as homotopy groups of suitable function-spaces.

The material on homotopy groups may be found in *Hilton (1) chaps. 2, 4 and 5, Hu chaps. 4 and 5 or Spanier chap. 7. For the more general groups, see Eckmann.

At this point, or perhaps earlier, it becomes evident that one needs methods for effective calculation. This means homology theory. To give the student the feel of the subject, one should probably begin with finite simplicial homology theory. It is enough to consider only finite simplicial complexes equipped with a given ordering of the vertices; this cuts out a good deal of confusing verbiage about orientations. It is necessary to give the basic definitions and certain variations of them: relative homology, cohomology, and the use of different coefficient groups. It is not necessary to prove the topological invariance of finite simplicial homology; students at this stage usually find the proof tedious and unilluminating, and in any case the result follows from later theorems.

The material on finite simplicial homology may be found in the Séminaire H. Cartan 1948/49 (2nd ed.) exposés 1-4, *Hilton and Wylie chaps. 2 and 5, Hocking and Young chaps. 6 and 7 or Spanier chap. 4.

Next one must introduce the Eilenberg-Steenrod axioms, set up singular homology theory, and prove that it satisfies the axioms. Here it is open to argument whether one should set up both the theory based on simplexes and the theory based on cubes, or whether one should use only simplexes. The arguments in favour of cubes are as follows. First, it may be held that the student gains from seeing that there are at least two ways of setting up a homology theory, and that any way will do providing that it works. Secondly, there are various points at which is is marginally easier or more convenient to work with cubes rather than simplexes, and at such points it is pleasant to be able to mention cubes. (Such points arise, for example, in passing from a geometrical homotopy to a chain homotopy, and in proving the Hurewicz isomorphism theorem.) Thirdly, the cubical theory is used in various classical papers which the student might want to read, such as Serre's thesis (see §5). The arguments against cubes are as follows. Against the second and third points, it appears to be true that by using extra effort, or later methods, it is possible to avoid the use of cubes at all points where they are easier or were used by classical authors. And therefore, against the first point, why spend the time and risk confusing the issue? Personally, I still like cubes. In any case, at this stage it is certainly not necessary to prove the equivalence of the two singular theories, or that the singular theories agree with the finite simplicial theory on finite simplicial complexes; both results follow from later theorems. However, one should carry the work far enough to compute the homology of a few simple spaces such as spheres.

There are many good accounts available of this material. They include Eilenberg and Steenrod chaps 1 and 7, *Greenberg part 2, Hilton and Wylie chap. 8, Spanier chap. 4 and Wallace chaps. 5, 6, 7 and 8. The original paper by S. Eilenberg ('Singular homology theory', Annals of Mathematics 45 (1944), 407-447) is as pleasant to read as any, and is recommended; but with the other sources available it would be hard to justify reprinting 40 pages. I have however found space for the original paper by Eilenberg and Steenrod (Paper no. 2) which is both elegant and lucid.

The final topic which should be included in a first course is the Hurewicz isomorphism theorem. Technically, of course, it is possible to delay the proof until further machinery is developed and one can give the painless proof due to Serre (see §8). Personally I prefer to give a fairly elementary proof at this stage. Such a proof has two main pillars: the additivity lemma, and the result that the homology of an n-connected space X can be defined in terms of singular simplexes or cubes with their n-faces at the base-point. If one uses cubes, the additivity lemma can be proved fairly easily by direct geometrical construction; alternatively, one can prove everything at once by induction over the dimension. If one uses simplexes, it is still possible to prove the additivity lemma by direct geometrical construction, but the proof is unpleasant, and in my opinion the proof by induction is preferable. The homology result is straightforward, but at this stage probably involves an irreducible amount of work, which is worse for cubes because of normalisation. (The work can be made easier if one has available the geometrical realisation of the total singular complex of X - see §3. This singular complex may be either simplicial or cubical; its 'realisation' is a CW-complex possessing a map to X, and this map can be deformed in the required way by standard theorems. However, one would not expect this 'realisation' to be available at this stage.)

Proofs of the Hurewicz isomorphism theorem are given in *Spanier chap. 7 and G. W. Whitehead chap. 2.

This completes the material appropriate to a basic course, except that some authorities would include some of the material which I have collected for convenience in §4. From this point on there is much more freedom about the order in which the material can be taken. In fact, the ordering of the sections below does not reflect the order in which I hope a student would learn the subject. Sections 6 and 7 are placed where they are because of their close

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relation with §5; but I would hope that a student would learn something from §§10 and 12 at an early stage.

2. Categories and functors

The student cannot escape learning about these as he goes along. Thus no special reading is necessary. If references are required, see Eilenberg and Steenrod chap. 4, Freyd, MacLane chap. I, Mitchell or *Spanier chap. 1.

3. Semi-simplicial complexes

The student should know the basic definitions; these may be found in Hilton and Wylie pp 358-359, Hu pp 140-142 or *MacLane pp 233-236. (The theory is taken rather further in the Séminaire H. Cartan, 1956/57, exposé 1.) These complexes are useful in formalising some of the constructions and proofs about singular homology. They are also valuable in homological algebra; here they allow one to start from strictly algebraic or combinatorial foundations, and yet obtain objects to which one can apply all the techniques of algebraic topology (see for example André). Personally, I am not too much impressed by the arguments that they provide a good category in which to do homotopy theory, although they have been much used in discussing Postnikov systems (see §10). The use of these complexes seems most profitable when one can consider semi-simplicial complexes with a strong algebraic structure. This subject is well represented by Milnor's paper 'On the construction FK' reprinted here as Paper no. 10. See also Bousfield, Curtis, Kan, Quillen, Rector and Schlesinger, 'The mod p lower central series and the Adams spectral sequence', Topology 5 (1966), 331-342.