Introduction

In the summer of 1968 I received an invitation to give a series of lectures on some topic in algebraic topology at the University of São Paulo, Brazil. The level was to be approximately that of a second year graduate course (at Cornell University, for example); that is to say, the audience would consist of people versed in ordinary homology and cohomology theory and in homotopy theory. It further appeared that the course would consist of nine lectures, each of one and a half hours' duration, so that the topic had to be one in which a satisfactory 'pay-off' could be achieved in a relatively short time.

The topic of general cohomology theory, with special reference to K-theory, suggested itself very naturally. The existence of a comprehensive and readable literature, both original papers and books, made it justifiable to highlight some of the more sensational achievements of K-theory without giving details of all proofs.* Moreover, by presenting K-theory as just one - though, to be sure, one of the most exciting - of the 'new' cohomology theories, it was natural to give some attention to properties of general cohomology theories and their relation to ordinary cohomology. The study of the entire category of cohomology theories and their interrelations is itself an area of active research and I was

* The following two books constituted the basic reference works for the course, as they do for these notes: M. F. Atiyah, K-theory, Benjamin; D. Husemoller, Fibre Bundles, McGraw-Hill.

particularly concerned to place my audience in the position of being able to tackle open questions in this area; on the other hand, K-theory itself is now a well-established tool in topology and is probably not itself a promising topic for research among new initiates.

The leitmotif of the course was the presentation of two connections between ordinary and general cohomology, namely, the generalized Atiyah-Hirzebruch spectral sequence and the character. Specializing to (complex) K-theory, one obtains the Atiyah-Hirzebruch spectral sequence and the Chern character and these two tools were used to prove Adams' celebrated theorem on the non-existence of elements of Hopf invariant 1 in $\pi_{2n-1}(S^n)$, $n \neq 2, 4, 8$. Since Adams' original proof involved the development of the subtle Adams spectral sequence relating cohomology to stable homotopy and a deep study of secondary cohomology operations (Adams' final paper in Annals of Mathematics occupied 80 pages!), it is a signal triumph of K-theory to produce a proof which only requires a knowledge of certain very natural primary operations in K-theory. Applications of Adams' theory were given to obtain results in classical linear algebra.

A more detailed description of these course notes is as follows. In Chapter I we define general cohomology theories, both reduced and unreduced, and list certain elementary results. A feature of the presentation here is that the theory is presented as an absolute theory and the corresponding relative theory is deduced from it. This makes for a great gain in simplicity and reduces the number of axioms to three, the suspension axiom replacing the usual coboundary homomorphism; on the other hand, the more usual (relative) axiom system may be more suitable for certain categories of topological spaces (e.g. compacta). The important notion of a representable theory is described in this chapter.

In Chapter II a general theory of spectral sequences is developed. This chapter could be largely ignored (except insofar as it establishes notation) by anybody familiar with spectral sequence theory. The treatment is based on that of Eckmann-Hilton [Exact couples in an abelian category, Journ. of Algebra (1966), pp. 38-87], but is considerably more elementary in that only abelian groups are considered and very explicit constructions are made of the various groups and homomorphisms which arise in the functorial passage from exact couples to spectral sequences. It is not at all claimed that this more elementary approach is conceptually simple; however, it appeared at the time that the audience was unfamiliar with more abstract categorical reasoning, and these notes, intended as a faithful record of the course, reflect the audience's preference. The reader who prefers the more categorical approach should consult the paper by Eckmann-Hilton. Not everything in this chapter is directly applicable to the material of Chapter III, but the structure of the chapter was adapted to the purpose of this very particular application.

Chapter III describes the generalized Atiyah-Hirzebruch spectral sequence. For a given finite-dimensional complex X this spectral sequence passes from ordinary cohomology $H^p(X;\check{h}^{q-p})$ with coefficients in the $(q-p)^{th}$ component of the coefficients of the theory h, converging to the graded group associated with $h^q(X)$, suitably filtered. The filtration is given by $F^p h^q(X) = \ker(h^q(X) \to h^q(X_{p-1}))$, where X_{p-1} is the (p-1)-skeleton of X. Moreover, the convergence is finite and the filtration is finite: precisely if $\dim X = k$, then $E_k = E_{k+1} = \ldots = E_\infty$ in the spectral sequence and $F^{k+1}h(X) = 0$, $F^0 h(X) = h(X)$. Various conclusions are drawn from the spectral sequence; in particular it is shown that if Q is the group of rationals then

$$h^n(X) \otimes Q = \bigoplus_{p+q=n} H^p(X;Q) \otimes \check{h}^q .$$

The natural transformation $h \to h \otimes Q$ is called the <u>character</u> of the theory h. As the formula above shows, $h \otimes Q$ depends only

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on ordinary cohomology (with rational coefficients) and the torsion-free part of the coefficient group of h.

In Chapter IV K-theory is described using vector bundles over the given space X. Since all the emphasis here is on complex K-theory, the vector bundles are themselves complex, but, of course, real vector bundles are also presented. It is shown how the Grothendieck group K(X) of equivalence classes of vector bundles over X breaks up naturally into a direct sum $\tilde{K}(X) \oplus Z$, the second component being the dimension of the 'fibre' over the base point. Then $\tilde{K}(X)$ may be identified with the set of based homotopy classes of maps, $X \to B_U \times Z$, where B_U is the classifying space for the 'big' unitary group U. Then Bott periodicity, asserting a homotopy equivalence between ΩU and $B_U \times Z$, leads to the definition of the reduced cohomology theory $\tilde{K}^n(X)$ and the free cohomology theory $K^n(X)$. Moreover, the tensor product of bundles leads to a commutative ring structure in K(X) with $\tilde{K}(X)$ as an ideal. The Grothendieck group is particularly simple in this case due to the fact that, to any bundle ξ, there exists a bundle η such that the Whitney sum $\xi \oplus \eta$ is a trivial bundle. It follows easily from this that $\tilde{K}(X)$ is naturally isomorphic to the set of stable equivalence classes of vector bundles over X.

In Chapter V the Chern character is defined using the classical theory of Chern classes for a complex vector bundle and it is identified with the character defined in Chapter III from the spectral sequence. The Adams operations in K-theory are introduced and their relation to the Chern character is elucidated with the aid of the so-called splitting principle; this principle asserts that, given any vector bundle ξ over X there is a space X_ξ and a map $f: X_\xi \to X$ such that f^* is a monomorphism in ordinary integral cohomology (and therefore also in cohomology over Q) and $f^*\xi$ is a sum of line-bundles. Properties of the Chern character in the special case where the homology of X is torsion-free, together with the Adams operations, are then used to prove the celebrated Adams theorem on the non-existence of elements of Hopf invariant 1. The actual theorem proved asserts that if X is a finite complex such that $H^*(X;Z) = Z[a]/a^3$ with $\dim a = n$, then $n = 2, 4$ or 8. Consequences of this theorem then close the chapter.

The course itself culminated in a discussion of the problem of extending cohomology theories from certain categories of topological spaces to larger categories. The material in this final section of the course was a digest of an article which has recently been published, namely: Peter Hilton, On the construction of cohomology theories, Rend. di Matem. (Roma), 1968, 219-232. This article, which was itself based on a talk given at the Istituto Matematico, Universita di Roma, dealing with the work of A. Deleanu and the author, is here reproduced as an appendix to these notes, by kind permission of Professor B. Segre.

It should be emphasized that this article only initiates the discussion of the extension problem and leaves open many questions. It is thus to be hoped that further work along these lines will prove fruitful. Of course, other topics would have served equally well (or better) to round off the course and provide the audience with the stimulus to undertake original work on general cohomology.

Many acknowledgements are due. I am indebted to my friend Frank Adams for the inspiration provided by his seminal work on general cohomology theories, K-theory, the Hopf invariant problem, the vector fields problem, and a host of other major contributions to algebraic topology. I am indebted to my friend Beno Eckmann for allowing me to borrow freely from the notes of the course, Cohomologie et Classes Caractéristiques, which he gave at a symposium held under the auspices of C. I. M. E. in 1966. I am indebted to my friend Aristide Deleanu for allowing me to publish the appendix. I am indebted to my friends and colleagues Carlos de Lyra and Renzo Piccinini for preparing these notes with great

care, skillfully establishing order out of my somewhat informal presentations; they are in no way to be held responsible for the residual imperfections of this text. Above all, however, I am grateful to the Instituto de Pesquisas Matematicas for inviting me to São Paulo to give this course, to Professor de Lyra for the fine way in which he made arrangements to secure the best possible conditions for my visit, and to the many friends and colleagues, acquired during those very pleasant weeks, whose enthusiasm for mathematics was only matched by their kindness and hospitality.

Peter Hilton

Cornell University, Ithaca, New York
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Note added in proof.

Since the preparation of these notes, further progress has been made in connection with the theory described in the Appendix. This progress is reported in the following articles:

A. Deleanu and P. J. Hilton, 'On the generalized Čech construction of cohomology theories', Battelle Research Report No. 28, 1969.

A. Deleanu and P. J. Hilton, 'On extensions of cohomology theories and Serre classes of groups', Battelle Research Report No. 34, 1970.

1. General Cohomology Theories

We shall denote by $\mathscr{C}$ the category of finite-dimensional cell-complexes with base point; the morphisms are base-point preserving maps from one based space to another. Furthermore, homotopies will be assumed to preserve base-points.

For the definition of a general cohomology theory, we need the (reduced) suspension functor in $\mathscr{C}$, a covariant functor Σ from $\mathscr{C}$ to $\mathscr{C}$ defined as follows. Denote by I the unit interval $[0, 1]$ and let $\dot{I}$ be the subspace $\{0, 1\}$ of I. Let $(X, x_0) \in \mathscr{C}$; then ΣX is defined to be the quotient space $X \times I/(X \times \dot{I} \cup x_0 \times I)$ with the obvious base-point. If $f:(X, x_0) \to (Y, y_0)$ is a map of $\mathscr{C}$, define $\Sigma f : \Sigma X \to \Sigma Y$ by observing that the map $(x, t) \to (f(x), t)$ is compatible with passage to the quotient and thus induces a map Σf.

We also need the construction known as the mapping cone. Let $CX = X \times I/(X \times 1 \cup x_0 \times I)$ be the reduced cone on X with X embedded in CX by $x \to (x, 0)$; and let $f: X \to Y$ be a map of $\mathscr{C}$. Define the mapping cone C_f of f as the quotient space of the topological sum $CX + Y$ by the following identifications: x is identified to $f(x) \in Y$ for all $x \in X$. In particular, if $u: X \to \{x_0\}$ is the constant map, C_u is just ΣX. There is an evident embedding $i: Y \subseteq C_f$. Also if $f: X \to Y$ is an inclusion of a subcomplex, there is a natural homotopy equivalence $C_f \to Y/X$.

Let $X, Y \in \mathscr{C}$ have base-points x_0 and y_0 respectively. The spaces $X \vee Y$ and $X \wedge Y$ are defined as follows: $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ with base-point (x_0, y_0) and topology given by the natural imbedding $X \vee Y \subset X \times Y$; the space $X \wedge Y$ is defined to be the quotient space $X \times Y / X \vee Y$. The operations $\vee$ and $\wedge$ are associative and commutative in the category $\mathscr{C}$;

furthermore $\wedge$ is distributive with respect to $\vee$ (Spanier, E.: Function Spaces and Duality, Annals of Math. 70 (1959); 338-378).

In particular, if $X = S^1$, the standard unit 1-sphere with base-point, we have the natural homeomorphism $S^1 \wedge Y \approx \Sigma Y$. It is also not difficult to check that $S^n \wedge S^m \approx S^{n+m}$. By iteration one gets $S^n \wedge Y \approx \Sigma^n(Y)$.

A (reduced) cohomology theory is a family $h = \{h^n; n \in Z\}$ of contravariant functors from the category $\mathscr{C}$ to the category of abelian groups and homomorphisms, together with a family of natural transformations $\sigma = \{\sigma^n; n \in Z\}$, $\sigma^n : h^n \to h^{n+1}\Sigma$ subject to the following axioms:

(1.1) (Homotopy axiom) If $f \simeq g$, then $h^n(f) = h^n(g)$ for every $n \in$ (or, as we may write, $f^* = g^*$);

(1.2) (Suspension axiom) σ^n is a natural equivalence for every $n \in Z$;

(1.3) (Exactness axiom) Given $X \xrightarrow{f} Y \xrightarrow{i} C_f$, the sequence of abelian groups and homomorphisms
$$h^n(X) \xleftarrow{f^*} h^n(Y) \xleftarrow{i^*} h^n(C_f)$$
is exact for every n.

Associated to every map $f: X \to Y$, a long exact sequence arises as follows. Apply axiom (1.3) successively to the sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{j} C_i \simeq \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \dots ,$$

which arises using the homotopy equivalence $C_i \simeq C_f / Y = \Sigma X$. We get an exact sequence (using (1.1) and (1.2))

$$\begin{array}{ccccccccc} h^n(X) & \xleftarrow{f^*} & h^n(Y) & \xleftarrow{i^*} & h^n(C_f) & \xleftarrow{j^*} & h^n(\Sigma X) & \xleftarrow{(\Sigma f)^*} & h^n(\Sigma Y) \\ & & & & & & {\scriptstyle \sigma^{n-1}} \uparrow \cong & & \uparrow \cong \\ & & & & & & h^{n-1}(X) & \xleftarrow{f^*} & h^{n-1}(Y) \dots \end{array}$$

or

(1.4) $$\dots \longleftarrow h^n(X) \xleftarrow{f^*} h^n(Y) \xleftarrow{i^*} h^n(C_f) \xleftarrow{\delta} h^{n-1}(X) \xleftarrow{f^*} h^{n-1}(Y) \longleftarrow \dots ,$$
where $\delta = j^* \sigma^{n-1}$.

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If the map $f:X \to Y$ is an imbedding, we define $h^n(Y, X) = h^n(C_f) = h^n(Y/X)$. Using the isomorphism $h^n(\Sigma X) \cong h^{n-1}(X)$ of axiom (1.2), we get the expected long exact sequence of the pair (Y, X):

$$(1.5) \quad \ldots \longleftarrow h^n(X) \longleftarrow h^n(Y) \longleftarrow h^n(Y, X) \longleftarrow h^{n-1}(X) \longleftarrow \ldots$$

Thus our axiom system implies the usual Eilenberg-Steenrod axioms except for the dimension axiom (see definition below).

As a first example of a general cohomology theory we have the usual cohomology theory H; in fact, this theory satisfies one more axiom besides axioms (1.1), (1.2) and (1.3), namely the dimension axiom, which asserts that for a 0-sphere S^0, $h^n(S^0) = 0$ if $n \neq 0$. Actually, if a cohomology theory satisfies also the dimension axiom, then we have uniqueness in the category $\mathscr{C}_f$ of finite cell-complexes. That is, there is only ordinary cohomology theory (with specified coefficients). We will give a proof of this later.

We list a few more examples of cohomology theories.

(1) Given any theory h and an integer k, a new theory ${}^k h$ can be trivially obtained by setting $({}^k h)^n = h^{n+k}$. This is called 'suspending the theory h'.

(2) Given any space $Z \in \mathscr{C}$ and a theory h, we define a new theory h_Z by setting $h^n_Z(X) = h^n(X \wedge Z)$, for every $X \in \mathscr{C}$, $n \in Z$.

To verify axiom (1.2), recall that $\Sigma X = S^1 \wedge X$ and use the associativity $S^1 \wedge (X \wedge Z) \approx (S^1 \wedge X) \wedge Z = (\Sigma X) \wedge Z$. Axiom (1.3) is satisfied because, given $f:X \to Y$, the spaces $C_f \wedge Z$ and $C_{f \wedge 1}$ are homeomorphic, where $f \wedge 1:X \wedge Z \to Y \wedge Z$. Using this construction, new non-trivial examples of cohomology theories may be obtained.

(3) Given any theory h, we define theories $\bar{h}$ and $\bar{\bar{h}}$ by

$$\bar{h}^i = \bigoplus_{n \in Z} h^n \quad \text{(direct sum)}$$

$$\bar{h}^i = \prod_{n \in Z} h^n \quad \text{(direct product)}$$

for all i. Since direct sums and products preserve exactness, one verifies easily that this is a cohomology theory.

(4) We sketch now an important example of a cohomology theory: stable cohomotopy theory. Its origin is a theorem of K. Borsuk, which asserts that if X is a space of dimension $\leq 2n - 2$, the set $[X, S^n]$ of all homotopy classes of maps $X \to S^n$ has an abelian group structure; furthermore, one shows that the group $[X, S^n]$ is isomorphic to all the groups $[\Sigma^q X, S^{n+q}]$, $q \geq 0$, the isomorphisms being given by the suspension map $[f] \to [\Sigma f]$.

Consider a finite-dimensional CW-complex X and an integer k. Consider (for $k \geq 0$) the sequence

$$[X, S^k] \xrightarrow{\Sigma} [\Sigma X, S^{k+1}] \xrightarrow{\Sigma} \dots \longrightarrow [\Sigma^q X, S^{k+q}] \longrightarrow \dots$$

or, for $k < 0$, the sequence

$$[\Sigma^{-k} X, S^0] \longrightarrow (\Sigma^{-k+1} X, S^1] \to \dots \longrightarrow [\Sigma^{-k+q} X, S^q] \longrightarrow \dots$$

In either case, these sequences become sequences of abelian groups and homomorphisms and they stabilize; we write $\Pi^k(X)$ for the stable value. The family $\{\Pi^k; k \in Z\}$ defines a cohomology theory. Axioms (1.1) and (1.2) are trivially verified; as for axiom (1.3), it is enough to observe that given a sequence $X \xrightarrow{f} Y \xrightarrow{i} C_f$ and a space Z, the homotopy extension property shows the exactness of the sequence of sets $[C_f, Z] \to [Y, Z] \to [X, Z]$. Exactness can be carried to the stable range since the direct limit preserves exactness* (Eilenberg, S. and Steenrod, N.: Foundations of Algebraic Topology, Princeton U. Press, Chap. VIII, Theorem 5.4).

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* This argument would only be needed if we allowed infinite-dimensional complexes X. For an N-dimensional complex X, $\pi^k(X) = [\Sigma^m X, S^{m+k}]$ for any $m \geq N - 2k + 2$.