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J. Aczel and J. Dhombres
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ENCYCLOPAEDIA OF MATHEMATICS AND ITS APPLICATIONS

EDITED BY G.-C. ROTA

Volume 31

Functional equations in several variables

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Dedicated to
Beckie, Cathy, Julie, Pascale, Robbie, Sylvestre and Thomas

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PREFACE

Functional equations are equations in which the unknown (or unknowns) are functions (but we shall not cover such a large domain). As the title shows, we deal in this book with 'functional equations in several variables'. This does not mean that we consider only equations in which the unknown functions are of several variables (in other words, multiplace functions) but, rather, that there are several variables *in the equation*. Cauchy's equation,

$$f(x+y)=f(x)+f(y),$$

containing *two* variables x, y and one unknown function f of *one* variable, is a classical example. On the other hand, iterative equations, differential, difference–differential, difference, integral and similar equations contain (in most cases) just one variable, if the unknown function is of one variable (a one-place function). We will deal here with *functional equations in which the number of variables is greater than the number of places in the unknown function* (or, if there is more than one unknown function, greater than the number of places in the unknown function with the smallest number of places).

There are several thousand works on functional equations, even in this restricted sense, and it is impossible to summarize their contents in a book of this size. So we have tried to make the *bibliography* encyclopaedic (although not complete) – it is organized by years. The reader is encouraged to refer to as many of the works listed in the bibliography as possible. (Some are quite elementary.) An amazing number of rediscoveries can be spotted in this way.

We felt it might be useful to the reader for us to include a historical chapter at the end of the book (Chapter 21). This chapter can be read *first* in order to get a bird's eye view (which makes a longer introduction unnecessary), ignoring or checking the few references to other sections in it. But, also, we think that this last chapter can be read again with profit after the first twenty chapters have been studied.

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From this historical chapter, the reader will see that a somewhat general theory of functional equations in several variables is a relatively recent development. It can also be seen that, from their very beginnings, functional equations arose from applications, were developed mostly for the sake of applications and, indeed, were applied quite intensively as soon as they were developed. Such a course of development is not typical in all theoretical aspects of mathematics. A characteristic example of this process can be found in the first, introductory, chapter which is devoted to the parallelogram law for vector addition.

This brings us to the natural question of motivation: *To what kind of problems can functional equations be applied* (and not only within mathematics). G.H. Hardy said that mathematicians are 'makers of patterns'. They often construct new notions, partly based on older ones, which may come from mathematics or from the natural, behavioural and social sciences, for instance from physics or economics. Thus the notion of affine vectors evolved in order to represent forces, the (homogeneous) linear function to describe proportionality, the logarithmic function to transform geometric sequences into algebraic ones, and the trigonometric functions to determine unknown parts of triangles from known ones. The next task of the mathematician is to describe the properties of these new objects, that is, to establish their relation to other objects in order to include them in a systematic and orderly way into an existing or new theory. (This is important because it is convenient to organize the notions in a pattern ruled by the same structure, with unity in the style of proofs.) The vectors, for instance, form the foundation of linear algebra, the trigonometric functions that of trigonometry. After stating the properties of the new objects (or at least some of them), what next?

If we are lucky, the properties deduced from the definitions of mathematical objects can be quite numerous, even too rich. According to the principle of economy of reasoning and to a legitimate desire for elegance, it is desirable to check whether the newly introduced objects are the *only ones* which have some of *the most important properties* that we have just established. It is here where functional equations may enter. One takes these properties as points of departure and tries to determine all objects satisfying them and, in particular, to find conditions under which there is unicity. This is done in the framework of the classical structures of mathematics, be they algebraic, order-theoretic, topological, or simply the real axis which carries all these structures. This is the fundamental procedure which motivates functional equations. We see that it is axiomatic in nature. But we see also that the axioms are not arbitrarily chosen, we do not generalize just for the sake of generalization. In fact, one prefers to take as points of departure those which are considered the most useful for applications. (Also practical concerns determine the procedure since we want to attain uniqueness.) Thus there is a feedback from the applications to the theory.

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This is not all. After such a uniqueness result is established, for instance that concerning the equation of the cosine (d'Alembert's equation), one also tries to deduce (using the equation, not the solutions) the classical properties of the cosine directly. One thus arrives at an elegant way of developing trigonometry which, with some changes, can also be used to introduce and develop elliptic and, especially, hyperbolic trigonometry. This procedure was followed (with varying degrees of clarity) by mathematicians in different ages, such as Oresme, Euler, d'Alembert, Abel, Cauchy, Lobachevskiĭ, Darboux, Picard, etc. Another famous example, outside the framework of our book, is the introduction of the gamma function through its functional equation, as developed by Artin.

As in all parts of mathematics, different authors' viewpoints and levels of discussion of functional equations are different. Also, sometimes the methods of reduction of one functional equation to another or the proof of their equivalence or discussion and/or reduction of regularity conditions, extension of domains, etc. give rise to further intrinsic developments. This is a further sign of the increasing maturity of this field of mathematics. On the other hand, the characterization of functions by their equations, as described above, involves many branches of pure and applied mathematics in the development of the theory of functional equations.

Our general intention in this book is to give the reader at least a general impression of what this subject is about by focusing on a relatively small number of examples, chosen with particular regard to applications but without neglecting theory. This explains why most technical connections among the different chapters are minimal, so that every chapter may constitute a unit of study in itself. Indeed, the reader can start with any chapter (and even many sections) with only occasional backward references. (Here the subject index at the end of the book will prove helpful.) An obvious exception is Chapter 2 which deals with Cauchy's equation and is intended to provide the reader with a certain number of definitions, techniques and results which will be used throughout the book (such as domain, extension of solutions, general solutions, regular solutions, etc.). However, from a more general mathematical point of view, connections between chapters can be found, and such connections can be used, for instance in both undergraduate and graduate courses or seminars. So can the several chapters and sections (or parts of them) devoted to connections with other parts of mathematics and to applications in and outside of mathematics. We give some examples below:

Combinatorics, probability and information theory	3; 5.4; 12; 16.4
Economics, decision making and mean values	7.3; 15; 17; 20
Almost periodic functions and harmonic analysis	5.3; 7.2; 8; 10.5

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Functional analysis, operator theory and Heaviside functions	6; 10; 11; 19.4; 19.5; 19.6
Geometry, nomography and physics	1; 4.2; 5.2; 6; 8; 9.1; 11.4; 18; 19.4
Groups, groupoids and semigroups	4.4; 5.3; 16.2; 16.3; 19.1; 19.2; 19.3
Trigonometric functions and number theory	1; 8; 13; 14; 16.3; 16.4; 16.5; 16.6.

(There are several more items related to these and other applications in the 'exercises and further results' at the end of different chapters.)

We also indicate some longer sequences which can be used for individual courses. The fundamental Chapter 2 should be added to each of them, and, clearly, some paragraphs of the following chapters and sections can be omitted if they do not fit into the subject of the course:

- 3; 4.1; 4.2; 4.4; 9.1; 11.2; 16.3; 16.5; 16.6; 19.1; 19.2; 20
- 5.1; 5.2; 5.3; 8; 10.5; 12; 13; 14
- 5.2; 6; 9.2; 11.3; 11.4; 14; 16.3; 16.4; 16.5; 16.6
- 10.4; 15; 16.4; 17; 18; 19.3
- 3; 6; 7; 10.2; 10.3; 11.2; 16.1; 16.2
- 5.1; 5.4; 6; 7.2; 10.1; 10.2; 19.4; 19.5; 19.6
- 1; 4.3; 4.4; 5.2; 7.2; 8; 11.1; 12; 13; 15; 16.2; 17; 18; 19.1; 19.2.

The pedagogical purpose of the book and the wish to at least indicate some further results and directions of research has led us to include more than 400 items as 'exercises and further results'. Some can be solved by a direct application of the material explained in the chapter in which the exercise appears; these exercises serve the usual aims of practice and of further applications of the results and proofs. Others are 'further results': extensions of the results described in the text to more general (algebraic, topological) settings and (more or less) related theorems. We sometimes give details of such generalizations and further results. Even where we do not generalize, the reader may want to try to prove such generalizations. Some 'exercises and further results' go deeper in the theory (and the most difficult ones are marked with a *). For some selected exercises and results we provide the reader with hints (which we hope won't become hindrances!), but we do not give hints to exercises which we consider to be easy or to results which are preceded by easily accessible references. In a large number of cases, these references are to the original contributions to the solution of the problem; in some other cases we preferred to give more modern or more easily accessible literature. On the other hand, in the main text, we have tried to be accurate concerning priorities, except for general background references for which we again wanted to quote easily accessible sources.

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Now we have to say a few words about the prerequisites which are necessary for a fruitful use of the book. We do try to avoid all unnecessary complications by limiting ourselves to simpler cases and we think that most chapters in the book can be read at the sophomore level, after mastering calculus, general and linear algebra, and perhaps basic Lebesgue theory of integration in $\mathbb{R}$ or $\mathbb{R}^n$. Incidentally, a previous monograph by one of the authors (Aczél 1966*c*) provides a more elementary approach to some parts of this book and also some further topics. Some applications require more specialized knowledge which we indicate there (some Banach algebra techniques for example, duality in functional analysis, and some measure theory). We try to explain at least the results we use. But, generally, no previous knowledge of the field to which functional equations are applied is required. This is true, for instance, in information theory, number theory, mean values, consensus allocations, geometric objects, almost periodic functions, relativity theory, etc.

While we have tried to 'homogenize' the presentation to some degree, the attentive reader will notice which parts were written by which author, due to their different styles (and mother tongues). Hopefully, this is not too terrible.

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Che Tat Ng has suggested several essential improvements and shortcuts. Gord Sinnamon helped checking the galley proofs. Richard Atkins and Gord Sinnamon helped finishing the subject index. Susan Aczél has compiled the author index and worked long and hard to put the bibliography together. Linda Gregory and Brenda Law have typed (in several versions), Dong Nguyen, Richard Atkins, Nicole Brillouët-Belluot and, last but not least, Bert Schweizer have read and corrected the manuscript and suggested improvements. Our warmest thanks are due to all of them and to colleagues, including anonymous referees, who have helped us with their comments. Support by the Natural Sciences and Engineering Research Council of Canada and the France-Canada Scientific Cooperation Agreement is gratefully acknowledged. We express also our thanks to the publishing and printing houses for their conscientious work.

Let us add that we welcome comments on the book or on any aspect of the subject.

János Aczél
Jean Dhombres

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References, numbering. The references in the bibliography are listed by years, within the same year alphabetically according to the authors' names, and works of the same author(s) in the same year are distinguished by letters. References are quoted in the text with the authors' names, the years, and, where necessary, with distinguishing letters. Theorems, propositions, lemmata, and corollaries are numbered consecutively in each chapter (so that Lemma 1 may be followed by Theorem 2 and that by Corollary 3) and formulas are separately numbered, also consecutively, within the chapters. In the same chapter they are referred to by these numbers, while in the references to another chapter the number of that chapter is attached, for example Theorem 2.3 or formula (3.5). (Some longer chapters are subdivided into sections but numbering of formulas, etc., is within chapters, not sections.) As usual, easier theorems are called 'Propositions'.

Exercises and further results at the end of the chapters are numbered (separately) and quoted in a similar way. Those with stars (*) are thought (by the authors) to be more difficult.

An *Author Index* and a *Subject Index* are, as usual, at the end. *Notations* are mostly standard, but we also summarize less standard notations at the end of the book.

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