1

Axiomatic motivation of vector addition

In this book we try to emphasize applications as motivation for functional equations. Of course we cannot present them in all exact details (as we try to do in the treatment of functional equations themselves). This first chapter is meant to give such a motivation for the Cauchy and d'Alembert equations. The reader who needs no convincing of their importance (or has any difficulty with Chapter 1) may proceed directly to Chapter 2.

One of the first problems to which functional equations were ever applied, and for whose sake functional equations were first solved, was the so called 'parallelogram of forces' or, in more modern language, the axiomatic motivation of the customary rule for addition of vectors (d'Alembert 1769; Poisson 1804, 1811; Picard 1928, pp. 4–17; Aczél 1966*c*, pp. 120–4, 1969*a*, pp. 7–11, 1976*a* are a few relevant references). Here we give a somewhat more satisfactory treatment, based on slightly weaker assumptions.

We start with the space V of all 3-tuples $\mathbf{p}=(x_1,x_2,x_3)$, $x_i \in \mathbb{R}$. This space is equipped with the euclidean distance

$$d(\mathbf{p},\mathbf{q})=((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2)^{1/2}$$

between $\mathbf{p}$ and $\mathbf{q}=(y_1,y_2,y_3)$. Our purpose is to characterize the additive law of such 3-tuples by geometric conditions.

Our fundamental tools will then be rotations on V, which are bijective transformations from V onto itself preserving $\mathbf{0}=(0,0,0)$ and distances in V. We recall that a rotation is completely determined by an axis through $\mathbf{0}$ and an angle, which is a real number modulo 2π. Given any two 3-tuples, $\mathbf{p}$ and $\mathbf{q}$, having the same positive distance from the origin, there exists one and only one rotation mapping $\mathbf{p}$ into $\mathbf{q}$. The angle of this rotation, say Θ, is the angle of the two 3-tuples, which can be determined by the analytic formula

$$\cos\Theta=\frac{x_1y_1+x_2y_2+x_3y_3}{(x_1^2+x_2^2+x_3^2)^{1/2}(y_1^2+y_2^2+y_3^2)^{1/2}},$$

where $\Theta\in[0,\pi]$ or $\Theta\in[\pi,2\pi]$ according to the orientation chosen on V (for example by choosing $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ to be a directed basis). The first will be called the smaller, the second the larger angle of the two vectors.

For the sake of simplicity, we call the 3-tuple

$$\mathbf{p}=(x_1,x_2,x_3)$$

the vector $\mathbf{p}$. We say that two vectors $\mathbf{p}=(x_1,x_2,x_3)$ and $\mathbf{q}=(y_1,y_2,y_3)$ have the same (or opposite) direction if there exist real numbers λ,μ of the same (respectively, opposite) sign such that

$$\lambda x_i=\mu y_i, \quad i=1,2,3.$$

We allow that one (but not both) of λ and μ be 0. For convenience we call $\mathbf{p}$ and $\mathbf{q}$ of the same direction in this case too. Naturally, the length of a vector $\mathbf{p}$ is its euclidean norm

$$|\mathbf{p}|=(x_1^2+x_2^2+x_3^2)^{1/2}.$$

With this in mind, we can state our theorem:

Theorem 1. *If, and only if, a binary operation $\circ$ on the set V of all vectors in the oriented three dimensional euclidean space*

(i) *is invariant under (really, covariant with) the rotations of the space, that is, the result of the operation (the 'resultant vector') undergoes the same rotation as the two factors ('components'),*
(ii) *is commutative and associative,*
(iii) *for two vectors of the same direction, $\circ$ reduces to arithmetic addition (the resultant vector also points in that direction and its length is the sum of the lengths of the components), and*
(iv) *the resultant of two vectors of equal length depends continuously upon their angle,*

then the operation $\circ$ is the usual vector addition (the resultant is obtained by the 'parallelogram rule').

Remark. Strictly speaking, suppositions like (i) and (iii) are themselves functional equations. However, for this introductory chapter we prefer the above verbal descriptions.

Proof. The most important consequence of (i) and of the commutativity (ii) is that *the resultant of two vectors of equal lengths lies in the plane spanned by them*, more exactly, *in the direction of the bisector of their angle* (using this direction as an axis for a rotation). Drawn from their intersection, the resultant could still lie either inside the smaller or the larger angle formed by the two vectors.

Condition (iii) assures that the zero-vector is a unit under $\circ$ ($\mathbf{p}\circ\mathbf{0}=\mathbf{0}\circ\mathbf{p}=\mathbf{p}$), and then again (i) shows that in $(V,\circ)$ *a vector* $\mathbf{p}$ *and a vector* $-\mathbf{p}$ *of the same length but opposite direction are inverses*: $(-\mathbf{p})\circ\mathbf{p}=\mathbf{p}\circ(-\mathbf{p})=\mathbf{0}$. Thus, with (ii), we have

(v) $(V,\circ)$ *is an abelian group with* $\mathbf{0}$ *as unit and* $-\mathbf{p}$ *as the inverse of* $\mathbf{p}$.

By (iii), the resultant of two vectors of the same length and of the same direction (angle 0) lies in the same direction, that is, in the direction of the bisector of their smaller angle (0). If there existed an angle of two vectors of the same length for which the resultant would lie in the bisector of their larger angle (Figure 1) then, by the continuity (iv), there would exist an angle, smaller than π, under which two vectors of the same length would have the zero-vector as resultant. But these would then be inverses to each other in $(V,\circ)$, while we have just seen that the inverse vectors always form angles π. (By (v) there exists just one inverse.) This contradiction shows that the resultant of two vectors of equal length always lies inside their smaller angle (and is the zero-vector if and only if the two vectors of equal and non-zero length have the angle π).

So we have determined the direction (in the bisector of their smaller angle) of the resultant of two vectors of equal length x. Keeping their angle fixed,

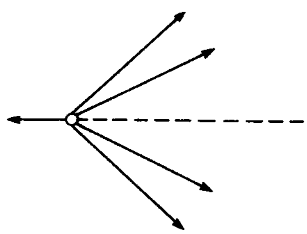

Fig. 1

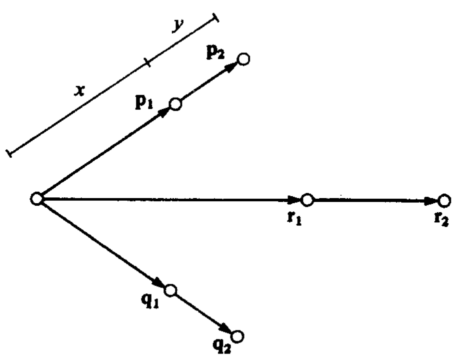

Fig. 2

we now determine its *length*, which we will denote by $g(x)$. This function g is defined on the set of all *nonnegative* numbers and is, in view of the above, nonnegative. Also (see Figure 2), if the vectors $\mathbf{p}_1, \mathbf{p}_2$ have the same direction, and also the vectors $\mathbf{q}_1, \mathbf{q}_2$, and ($|\mathbf{p}|$ denoting the length of the vector $\mathbf{p}$)

$$|\mathbf{p}_1| = |\mathbf{q}_1| = x, \quad |\mathbf{p}_2| = |\mathbf{q}_2| = y,$$

then, by (iii),

$$|\mathbf{p}_1 \circ \mathbf{p}_2| = |\mathbf{q}_1 \circ \mathbf{q}_2| = x + y$$

and, in view of the definition of g,

$$|\mathbf{r}_1| = |\mathbf{p}_1 \circ \mathbf{q}_1| = g(x), \quad |\mathbf{r}_2| = |\mathbf{p}_2 \circ \mathbf{q}_2| = g(y), \quad |(\mathbf{p}_1 \circ \mathbf{p}_2) \circ (\mathbf{q}_1 \circ \mathbf{q}_2)| = g(x + y).$$

But also, by (ii) and (iii), since $(\mathbf{p}_1 \circ \mathbf{q}_1)$ and $(\mathbf{p}_2 \circ \mathbf{q}_2)$ are collinear,

$$|(\mathbf{p}_1 \circ \mathbf{p}_2) \circ (\mathbf{q}_1 \circ \mathbf{q}_2)| = |(\mathbf{p}_1 \circ \mathbf{q}_1) \circ (\mathbf{p}_2 \circ \mathbf{q}_2)| = g(x) + g(y), \tag{1}$$

and so

$$g(x + y) = g(x) + g(y) \quad \text{for all nonnegative } x, y. \tag{2}$$

As mentioned in the introduction, this is Cauchy's equation (Cauchy 1821). Furthermore, as pointed out above,

$$g(x) \geqslant 0 \quad \text{for all nonnegative } x. \tag{3}$$

We will prove in Chapter 2 that (2) and (3) can hold if, and only if, there exists a nonnegative constant c such that

$$g(x) = cx \quad (x \geqslant 0).$$

The constant c has to be nonnegative, but, by (v), it cannot be 0 except when the two vectors of equal length have opposite directions. This shows that *the length of the resultant of two vectors of equal length and of angle different from* π *is proportional to the length of the components*. Here we have kept the angle of the two vectors constant. If we remove this restriction, then c will depend upon that angle.

We now determine the dependence of the length of the resultant of two vectors of equal length on their angle ($\neq \pi$). Because of what we have just proved, it is enough to take two vectors of unit length ('*unit vectors*'). Denote their angle, for convenience, by 2ϕ and the length of their resultant by $2f(\phi)$ (this is really our c, now dependent upon ϕ).

Take four unit vectors (Figure 3) $\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2$ ($|\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{q}_1| = |\mathbf{q}_2| = 1$) with angles $(\mathbf{p}_1, \mathbf{p}_2)\measuredangle = 2\psi = (\mathbf{q}_1, \mathbf{q}_2)\measuredangle$, $(\mathbf{p}_1, \mathbf{q}_1)\measuredangle = 2(\phi + \psi)$, $(\mathbf{p}_2, \mathbf{q}_2)\measuredangle = 2(\phi - \psi)$. Then $(\mathbf{p}_1 \circ \mathbf{p}_2, \mathbf{q}_1 \circ \mathbf{q}_2)\measuredangle = (\mathbf{p}, \mathbf{q})\measuredangle = 2\phi$ and (using (i)) $|\mathbf{p}_1 \circ \mathbf{p}_2| = |\mathbf{q}_1 \circ \mathbf{q}_2| = 2f(\psi)$. Also,

$$|(\mathbf{p}_1 \circ \mathbf{p}_2) \circ (\mathbf{q}_1 \circ \mathbf{q}_2)| = 2f(\psi) 2f(\phi) \tag{4}$$

(because the length of $\mathbf{p}_1 \circ \mathbf{p}_2$ or $\mathbf{q}_1 \circ \mathbf{q}_2$ is $2f(\psi)$, not 1 and the length of the resultant is proportional to the length of the components). On the other hand,

$$|\mathbf{r}_1| = |\mathbf{p}_1 \circ \mathbf{q}_1| = 2f(\phi + \psi), \quad |\mathbf{r}_2| = |\mathbf{p}_2 \circ \mathbf{q}_2| = 2f(\phi - \psi)$$

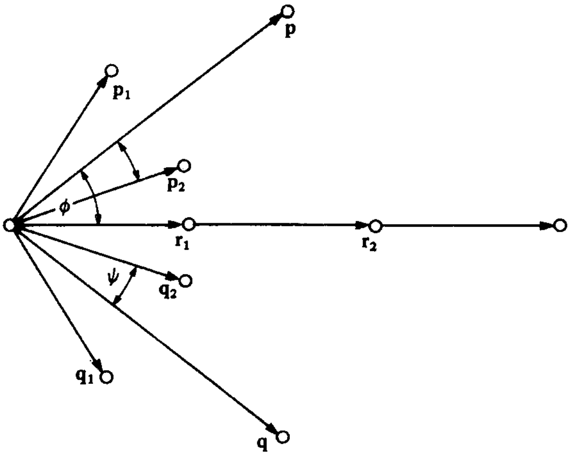

Fig. 3

and, by (4), (ii), (iii), and since $\mathbf{p}_1, \mathbf{q}_1$ and $\mathbf{p}_2, \mathbf{q}_2$ have a common bisector,

$$\begin{aligned} 4f(\phi)f(\psi) &= |(\mathbf{p}_1 \circ \mathbf{p}_2) \circ (\mathbf{q}_1 \circ \mathbf{q}_2)| = |(\mathbf{p}_1 \circ \mathbf{q}_1) \circ (\mathbf{p}_2 \circ \mathbf{q}_2)| \\ &= |\mathbf{r}| = |\mathbf{r}_1 \circ \mathbf{r}_2| = 2f(\phi+\psi) + 2f(\phi-\psi), \end{aligned} \tag{5}$$

or

$$f(\phi+\psi) + f(\phi-\psi) = 2f(\phi)f(\psi) \quad \text{whenever } 0 \leqslant \psi \leqslant \phi \leqslant \frac{\pi}{4}. \tag{6}$$

This is d'Alembert's functional equation.

By (iv), the function f, defined above on $[0, \pi/2]$, is *continuous*. As we will see in Chapter 8, a continuous function f satisfies (6) if (and only if) it has one of the following forms:

$$f(\phi) = 0, \quad \text{or } f(\phi) = \cosh C\phi, \quad \text{or } f(\phi) = \cos C\phi \tag{7}$$

for all $\phi \in [0, \pi/2]$ where C is a (real) constant. We may take $C \geqslant 0$, because (7) gives the same result for negative as for positive C. (We may also extend, if we wish, (6) and (7) to all real ϕ and ψ.) As we have seen, for vectors of equal lengths and the same or opposite directions,

$$f(0) = 1 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = 0$$

respectively. These exclude the first and second solutions enumerated (since cosh is nowhere 0) in (7) and specify

$$C = 2k+1, \quad \text{where } k \text{ is a nonnegative integer}, \tag{8}$$

in the third solution. But, as we have seen, by (v) the only vector which gives the zero vector as resultant with a given unit vector is the unit vector of

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opposite direction. So

$$f(\phi) \neq 0 \quad \text{if } 0 \leqslant \phi < \frac{\pi}{2}. \tag{9}$$

Therefore $C = 1$ in the third formula (7). Indeed, if $C = 2k + 1$ with $k > 0$, then $0 < \pi/[2(2k+1)] < \pi/2$ and $f[\pi/(2(2k+1))] = \cos[(2k+1)\pi/(2(2k+1))] = 0$, contrary to (9). Thus the only continuous solution of (6), appropriate for our purposes, is given by

$$f(\phi) = \cos\phi \quad \text{for all } \phi \in \left[0, \frac{\pi}{2}\right].$$

In conclusion, the length of the resultant of two unit vectors with the angle 2ϕ is $2\cos\phi$, the resultant of two vectors of equal length x lies in the bisector of their angle ($2\phi \in [0, \pi]$) and its length is $2x\cos\phi$. In other words, *for vectors of equal length, the parallelogram rule holds* (Figure 4), that is, if $|\mathbf{p}| = |\mathbf{q}| = x$ and $(\mathbf{p}, \mathbf{q}) \measuredangle = 2\phi \in [0, \pi]$, then $(\mathbf{p}, \mathbf{p} \circ \mathbf{q}) \measuredangle = (\mathbf{p} \circ \mathbf{q}, \mathbf{q}) \measuredangle = \phi$ and $|\mathbf{p} \circ \mathbf{q}| = 2x\cos\phi$.

The generalization of this result to vectors of *unequal* length can now be carried out by purely geometric considerations which we include here for completeness's sake.

First we consider *perpendicular* vectors $\mathbf{p}, \mathbf{q}$ (Figure 5). Take the diagonals of the rectangle spanned by $\mathbf{p}$ and $\mathbf{q}$ and draw parallels to one at the end points of $\mathbf{p}$ and $\mathbf{q}$, and at their common origin O, to the other. Thus we get two isosceles triangles. Denoting their sides, with the appropriate orientation (as in Figure 5), by the vectors $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{q}_1, \mathbf{q}_2$ (in addition to $\mathbf{p}$ and $\mathbf{q}$), we have by our previous results

$$\mathbf{p}_1 \circ \mathbf{p}_2 = \mathbf{p}, \quad \mathbf{q}_1 \circ \mathbf{q}_2 = \mathbf{q}, \quad \mathbf{p}_1 \circ \mathbf{q}_1 = \mathbf{0} \quad \text{(the zero vector)}$$

and, by obvious geometric facts (segments cut from parallel lines by parallel lines are equal), $\mathbf{p}_2 = \mathbf{q}_2 = \frac{1}{2}\mathbf{r}$, where $\mathbf{r}$ is the vector of the diagonal coming

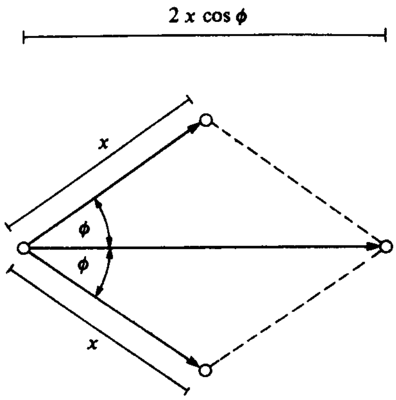

Fig. 4

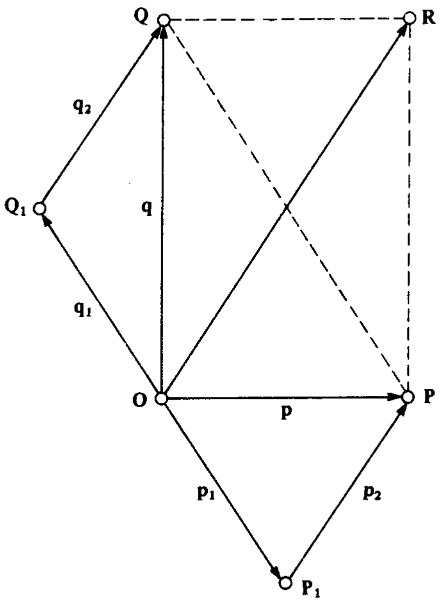

Fig. 5

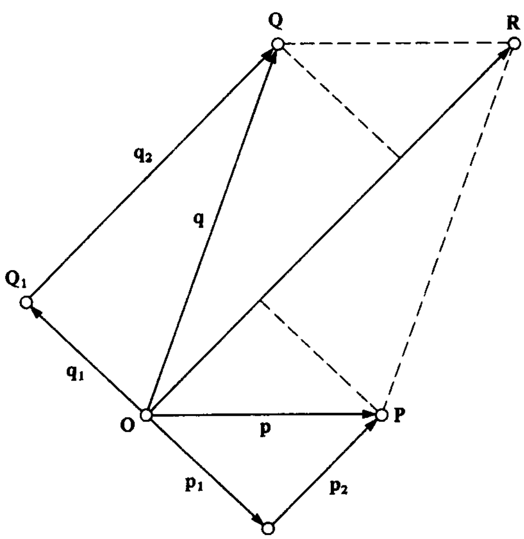

Fig. 6

from $\mathbf{0}$. Thus, cf. (ii) and (iii),

$$\mathbf{p}\circ\mathbf{q}=(\mathbf{p}_1\circ\mathbf{p}_2)\circ(\mathbf{q}_1\circ\mathbf{q}_2)=(\mathbf{p}_1\circ\mathbf{q}_1)\circ(\mathbf{p}_2\circ\mathbf{q}_2)=\mathbf{0}\circ(\mathbf{p}_2\circ\mathbf{q}_2)=\mathbf{r}, \tag{10}$$

that is, *the parallelogram rule holds also for perpendicular vectors.*

Finally (Figure 6), let $\mathbf{p}$ and $\mathbf{q}$ be completely *arbitrary* nonparallel vectors. Take the diagonal $\mathbf{r}=\overrightarrow{OR}$, starting from their common origin $\mathbf{0}$, of the parallelogram spanned by $\mathbf{p}$ and $\mathbf{q}$ and draw parallels to it from the end points of $\mathbf{p}$ and $\mathbf{q}$ and a perpendicular to it at $\mathbf{0}$. The sides of these two rectangular triangles, oriented as in Figure 6, will be denoted by the vectors $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{q}_1, \mathbf{q}_2$ (in addition to $\mathbf{p}$ and $\mathbf{q}$). By our previous results,

$$\mathbf{p}_1\circ\mathbf{p}_2=\mathbf{p},\quad \mathbf{q}_1\circ\mathbf{q}_2=\mathbf{q},\quad \mathbf{p}_1\circ\mathbf{q}_1=\mathbf{0},$$

and by (v), (iii), and obvious geometric considerations,

$$\mathbf{p}\circ\mathbf{q}=(\mathbf{p}_1\circ\mathbf{p}_2)\circ(\mathbf{q}_1\circ\mathbf{q}_2)=(\mathbf{p}_1\circ\mathbf{q}_1)\circ(\mathbf{p}_2\circ\mathbf{q}_2)=\mathbf{0}\circ(\mathbf{p}_2\circ\mathbf{q}_2)=\mathbf{r}. \tag{11}$$

The only case still not covered is that of two vectors $\mathbf{p}, \mathbf{q}$ of different lengths and opposite directions ((iii) takes care of any two vectors of the same direction). Let $\mathbf{p}$ be, say, longer than $\mathbf{q}$. Then $\mathbf{p}=\mathbf{r}\circ(-\mathbf{q})$ where $\mathbf{r}$'s length is $|\mathbf{p}|-|\mathbf{q}|$ and it has the same direction as $\mathbf{p}$. By (v),

$$\mathbf{p}\circ\mathbf{q}=[\mathbf{r}\circ(-\mathbf{q})]\circ\mathbf{q}=\mathbf{r}\circ[(-\mathbf{q})\circ\mathbf{q}]=\mathbf{r},$$

and so the parallelogram rule determines the resultant $\mathbf{p}\circ\mathbf{q}$ of two arbitrary vectors $\mathbf{p}, \mathbf{q}$. This concludes the proof of Theorem 1 (under acceptance of the regular solutions of (2) and (6)).

It is worth noticing that, in most of the proof, the associativity and commutativity contained in (ii) has been used – see (1), (5), (10) and (11) – in the form

$$(\mathbf{p}_1\circ\mathbf{p}_2)\circ(\mathbf{q}_1\circ\mathbf{q}_2)=(\mathbf{p}_1\circ\mathbf{q}_1)\circ(\mathbf{p}_2\circ\mathbf{q}_2). \tag{12}$$

This identity, called bisymmetry or *mediality*, is also an important functional equation, to which we will return later (Chapter 17).

Theorem 1 proves that the additive structure (a vector space) of the space V, equipped with a distance, is completely determined by its group of isometries, that is, by its unitary group. The same result is true for any n dimensional space, with $n\geqslant 3$, for the euclidean distance.

We shall give further results of this kind in Section 10.4.

The result of Theorem 1 is that $\mathbf{p}\circ\mathbf{q}=\mathbf{p}+\mathbf{q}$, where the right hand side, the 'sum' of the two vectors, is now defined by componentwise addition.

Exercises and further results

This was an introductory chapter, so only a few of the exercises are closely related to its subject matter; the rest deals with functional equations which can be handled directly, without use of the methods discussed in the rest of this book.

(For Exercises 1–3, cf. Aczél 1966*c*, pp. 24–5, 27–30.)

1. Show that the general solution $f:\mathbb{R}\to\mathbb{R}$ of

$$f(x+y)+f(x-y)=2f(x)\cos y \quad (x, y\in\mathbb{R})$$

is given by $f(x)=a\cos x+b\sin x$ $(x\in\mathbb{R})$, where a, b are arbitrary real constants. (As usual, the set of all real numbers is denoted by $\mathbb{R}$.)

2. Let E be a euclidean space of dimension $n>2$. Let $f:E\times E\to\mathbb{R}$ be invariant under any linear unitary transformation T of E ($Tx=A\cdot x$ where A is a matrix with determinant 1), that is, $f(Tx, Ty)=f(x, y)$ for x, y in E. Suppose also that f is linear in the first variable $[f(\alpha x+\beta z, y)=\alpha f(x, y)+\beta f(z, y)$ for all $\alpha, \beta\in\mathbb{R}$, $x, y, z\in E]$. Show that f is, up to a scalar multiple, the scalar product on E.

3. Let $\mathbb{R}^3$ be the usual euclidean space with three coordinates. Let $f:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ be such that, for any rotation T, $f(Tx, Ty)=T(f(x, y))$; $x, y\in\mathbb{R}^3$. Suppose that f is linear in the first variable. Show that f is, up to a scalar multiple, the vector product on $\mathbb{R}^3$.

4. (Aczél 1966*c*, pp. 223–4.) Find the general solution of the functional equation

$$F(x, y)+F(y, z)=F(x, z),$$

where x, y and z belong to a nonempty set S and $F:S\times S\to G$, where $(G, +)$ is a group.

5. Find the general solution $F:\mathbb{R}^2\to\mathbb{R}$ of the functional equation

$$F(x, y)F(y, z)=F(x, z), \quad (x, y, z\in\mathbb{R}).$$

6. (Moszner 1965.) Find the general solution $F:\mathbb{R}^2\to\mathbb{R}$ of the equation in Exercise 5 if we suppose only that the equation in the previous exercise holds for ordered triples $x\leqslant y\leqslant z$.

7. Show that, among $f:\mathbb{R}\to\mathbb{R}$, only constant functions satisfy

$$f\left(\frac{x+y}{2}\right)^2=f(x+y)f(x-y), \quad (x, y\in\mathbb{R}).$$

8. For any $x\in\mathbb{R}$, one defines the (leibnizian) *dilogarithm* by

$$\mathrm{Li}(x)=-\int_0^x\frac{\ln 1-t}{t}\,dt$$

and, for $x\neq 0$, $x\neq 1$,

$$M(x)=\mathrm{Li}(x)+\tfrac{1}{2}\ln|x(1-x)|.$$

Show that the following functional equation holds for $x\neq 1$, $y\neq 1$:

$$M(xy)=M(x)+M(y)+M\left(\frac{x}{x-1}(1-y)\right)$$
$$+M\left(\frac{y}{y-1}(1-x)\right)-2\lambda(x+y+xy)M(2),$$

where λ is the characteristic function of $[1, \infty[$ ($\lambda(x)=1$ if $x\geqslant 1$, $\lambda(x)=0$ if $x<1$).

9. (Aczél 1956*b*; Samuelson 1974; Christian 1983.) By forming the difference quotient and using the continuity and convexity of a^x, show that the derivative of a^x exists

everywhere and is proportional to a^x:

$$\frac{da^x}{dx} = a^x l(a).$$

Give a geometric motivation to

$$l(a^t) = tl(a) \quad \text{for all real numbers } t.$$

Show that there exists an exponential function whose derivative at 0 is 1. Call its base e. Then show that

$$l(s) = \log_e s = \ln s \quad \text{for all } s > 0.$$

(The above can serve as a definition of the natural logarithm.)

10. (Aczél 1966*c*, p. 26.) Determine all solutions $f:\mathbb{R}\to\mathbb{R}$ of

$$f(x+y) + f(x-y) = f(x)(y+2) - y(x^2 - 2y).$$

11. Suppose that $f:\mathbb{R}_+\to\mathbb{R}_+$ ($\mathbb{R}_+$ is the set of positive numbers) is continuous, strictly decreasing and satisfies

$$f(x+y) + f(f(x) + f(y)) = f[f(x+f(y)) + f(y+f(x))] \quad \text{for all } x, y\in\mathbb{R}_+.$$

Prove that $f(x) = f^{-1}(x)$ for all $x\in\mathbb{R}_+$ (f^{-1} is the inverse function of f).