1

The fundamental properties of CW-complexes

Balls and balloons are the standard models for the cells used in the theory of CW-complexes; thus, the chapter starts by 'playing' a bit with such toys. Next, it continues with a discussion of the problem of attaching n-cells to a space and with the actual construction of CW-complexes, followed by a detailed study of the fundamental properties of such spaces.

The unusual number given to the first section of this chapter, namely 1.0, stems from the fact that the material discussed therein is really very elementary.

1.0 Balls, spheres and projective spaces

The *ball* in the Euclidean space $\mathbf{R}^{n+1}$ is the space

$$B^{n+1} = \{s = (s_0, s_1, \ldots, s_n) : |s| \leqslant 1\};$$

its topological boundary is the *sphere*

$$\delta B^{n+1} = S^n = \{s \in B^{n+1} : |s| = 1\}$$

and the difference

$$\mathring{B}^{n+1} = B^{n+1} \backslash S^n$$

is the interior of the ball B^{n+1}, namely, the *open ball*. Observe that the ball $B^1 = [-1, 1]$ does not coincide with the unit interval $I = [0, 1]$ (in the sequel, the boundary of I will be denoted by $\dot{I}$).

Intuitively, one may view a sphere as the skin of a ball (i.e., a balloon). To blow up a balloon, there must be an opening, a 'base point'; thus, set the point $e_0 = (1, 0, \ldots, 0)$ as the base point of both B^{n+1} and S^n.

Spheres do not appear only as boundaries of balls; in addition to the inclusions

$$i^n : S^n \to B^{n+1},$$

it will be necessary to discuss several standard maps relating spheres and balls. The list of such maps described in this section is actually longer than that needed to develop the material herein. The primary two reasons are:

these maps could be used to fill in the details for the material sketched in the appendix;

some of the maps discussed could be used in the homology of cellular structures (e.g., the Hurewicz isomorphism theorem). Although homology is beyond the scope of this volume, it is a natural continuation for the theory here developed.

It is often convenient to view all balls B^{n+1} and all spheres S^n as contained in the space $\mathbf{R}^\infty$ of all sequences which vanish almost everywhere, via the embeddings $s \mapsto (s, 0, 0, \ldots)$; the topology of $\mathbf{R}^\infty$ is determined by the family of all Euclidean subspaces $\mathbf{R}^n$ (see Section A.2). Within this framework, consider the origin of $\mathbf{R}^\infty$ as the 0-*ball*

$$B^0 = \{0\},$$

whose boundary is the 'sphere'

$$\delta B^0 = S^{-1} = \varnothing,$$

and which coincides with its interior

$$\mathring{B}^0 = B^0.$$

In contrast with these 'minimal' models B^0 and S^{-1}, one has the *infinite ball* $B^\infty = \bigcup_{n \geq 0} B^n$ and the *infinite sphere* $S^\infty = \bigcup_{n \geq 0} S^n$ as subspaces of R^∞. Notice that these two infinite models are determined by the corresponding families of finite models (see Corollary A.2.3).

The ball B^n is embedded into the ball B^{n+1} as a strong deformation retract; a suitable retraction is the map

$$j^n : B^{n+1} \to B^n,$$

given by

$$j^n(s) = (s_0, \ldots, s_{n-1}).$$

Define the 'eggs of Columbus' using the map j^n, i.e. the inclusions

$$j_+, j_- : B^{n+1} \rightarrowtail B^{n+1}$$

given by

$$j_+(s) = (s_0, \ldots, s_{n-1}, \tfrac{1}{2}(s_n + \sqrt{1 - |j^n(s)|^2}))$$

and

$$j_-(s) = (s_0, \ldots, s_{n-1}, \tfrac{1}{2}(s_n - \sqrt{1 - |j^n(s)|^2})).$$

The function j_+ (resp. j_-) maps the upper (resp. lower) hemisphere onto itself and the lower (resp. upper) hemisphere onto the equatorial ball B^n (see Figure 1).

The deformation

$$d^n : (B^n \times B^n) \times I \to B^n \times B^n$$

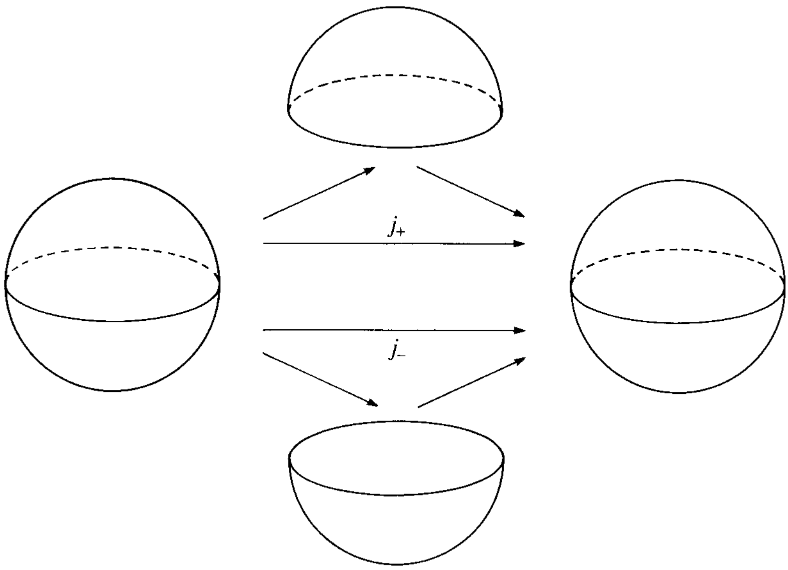

Figure 1

defined by

$$d^n((s,s'),t) = ((1-t)s + t(\tfrac{1}{2}(s+s')), (1-t)s' + t(\tfrac{1}{2}(s+s'))),$$

for every $(s,s') \in B^n \times B^n$ and every $t \in I$, shows that the diagonal subspace $\Delta B^n \subset B^n \times B^n$ is a strong deformation retract of $B^n \times B^n$; thus, balls are LEC spaces (see Section A.4, page 253).

The sphere S^{n-1} is included into the sphere S^n as its equator, and this inclusion, in turn, extends to embeddings

$$i_-, i_+ : B^n \rightarrowtail S^n$$

of the ball B^n into the southern, respectively northern hemisphere of S^n, given by

$$i_-(s) = (s, -\sqrt{1-|s|^2})$$

and

$$i_+(s) = (s, \sqrt{1-|s|^2}),$$

respectively, having $j^n|S^n$ as common left inverse.

The maps i_-, i_+ are homotopic only in a very curious way; in fact, a homotopy can be constructed by observing that both maps are homotopic to the constant map onto the base point, but there is no homotopy between them relative to the boundary (see the end of this paragraph). Viewed as maps into B^{n+1} the maps i_-, i_+ are homotopic in a neat manner namely,

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rel. S^{n-1} via the map

$$h^n : B^n \times I \to B^{n+1},$$

given by

$$h^n(s,t) = (s, (2t-1)\sqrt{1-|s|^2}).$$

The importance of this map h^n resides in the fact that every homotopy rel. S^{n-1} given between two maps defined on B^n factors through h^n. In particular this shows: If i_-, i_+ were homotopic rel. S^{n-1}, any corresponding homotopy factored through h^n would yield a retraction of B^{n+1} onto S^n, contradicting Brouwer theorem (see TheoremA.9.4).

Next, recall that the map (Figure 2)

$$c^n : S^n \times I \to B^{n+1}$$

given by

$$c^n(s,t) = (1-t)e_0 + ts$$

induces a homeomorphism

$$S^n \wedge I \to B^{n+1}$$

where the symbol $\wedge$ denotes the usual *smash product*

$$S^n \wedge I = S^n \times I / S^n \times \{0\} \cup \{e_0\} \times I.$$

The formation of the smash product with one factor equal to I is also known as the *reduced cone construction*. The *reduced suspension* of a based space (X, x_0) is one step further away; this construction is given on the based space (X, x_0) by

$$\Sigma.X = X \times I / X \times \dot{I} \cup x_0 \times I$$

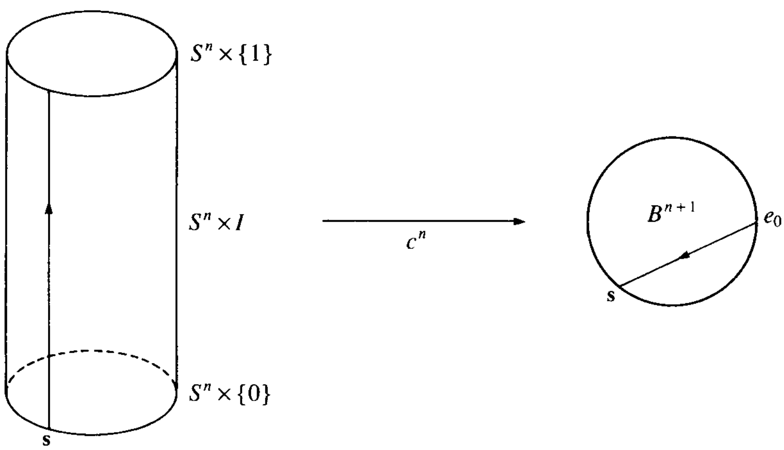

Figure 2

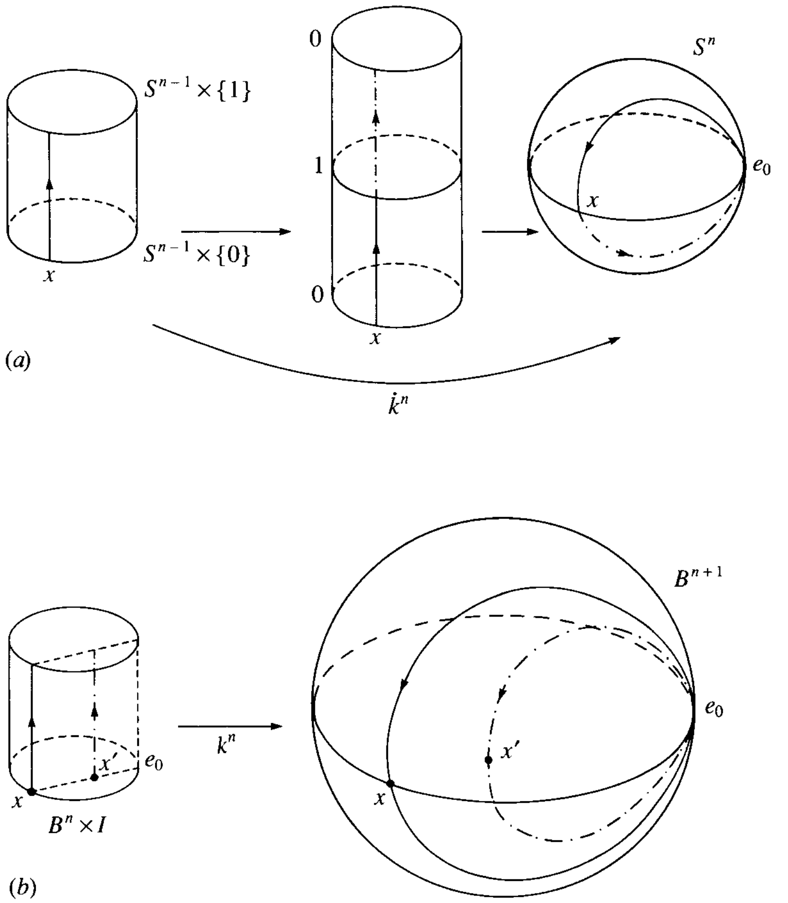

Figure 3

(note that $\Sigma.X$ is homeomorphic to the smash product $X \wedge S^1$); if $f:(Y, y_0)\to(X, x_0)$ is a based map, its suspension

$$\Sigma.f:\Sigma.Y\to\Sigma.X$$

is the map induced by $f\times 1 : Y\times I\to X\times I$.

For $n\geqslant 1$, define $\dot{k}^n : S^{n-1}\times I\to S^n$ (see Figure 3(*a*)) by

$$\dot{k}^n(s,t)=\begin{cases} i_+c^{n-1}(s,2t), & 0\leqslant t\leqslant \frac{1}{2}\\ i_-c^{n-1}(s,2-2t), & \frac{1}{2}\leqslant t\leqslant 1;\end{cases}$$

the map $\dot{k}^n$ takes $S^{n-1}\times \dot{I}\cup e_0\times I$ into e_0 and is bijective outside that space; thus, it induces a homeomorphism $\Sigma.S^{n+1}\to S^n$. Moreover, the map $\dot{k}^n$ can be extended to a map $k^n : B^n\times I\to B^{n+1}$ (see Figure 3(*b*))

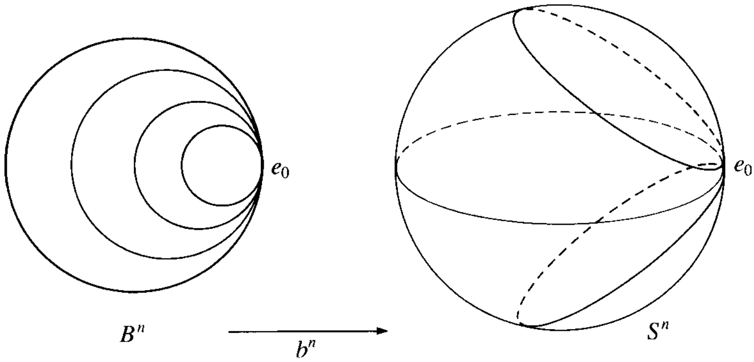

Figure 4

simply by taking

$$k^n(c^{n-1}(s,t'),t) = c^n(\dot{k}^n(s,t),t');$$

this latter map induces a homeomorphism $\Sigma.B^n \to B^{n+1}$. Finally, notice that the map $\dot{k}^n$ factors through the map c^{n-1}, and thus induces a map

$$b^n : B^n \to S^n;$$

formally, $b^n \circ c^{n-1} = \dot{k}^n$. In turn, the map b^n gives a homeomorphism between B^n/S^{n-1} and S^n. It is convenient to extend the definition of b^n to include $b^0 : B^0 \to S^0$ given by $b^0(B^0) = \{-1\}$. Figure 4 indicates that b^n is homotopic rel. $\{e_0\}$ to i_+ via a homotopy moving S^{n-1} only in the lower hemisphere.

The following maps are relevant to the definition of homotopy groups:

(i) the *units*

$$u^n : B^{n+1} \to B^{n+1}, \qquad \dot{u}^n : S^n \to S^n$$

defined for all $n \in \mathbf{N}$ as the constant-based maps;

(ii) the *inversions*

$$l^n : B^{n+1} \to B^{n+1},$$

defined by $l^n(k^n(s,t)) = k^n(s, 1-t)$ for every $(s,t) \in B^n \times I$; this inversion on B^{n+1} induces an inversion $\dot{l}^n : S^n \to S^n$ on S^n; notice that l^n, $\dot{l}^n$ are reflections about the hyperplane $\mathbf{R}^n \subset \mathbf{R}^{n+1}$:

$$l^n(s_0, \ldots, s_n, s_{n+1}) = (s_0, \ldots, s_n, -s_{n+1});$$

(iii) for $n \geqslant 1$, the *pinchings* (see Figure 5)

$$p^n : B^{n+1} \to B^{n+1} \vee B^{n+1}$$

given by

$$p^n(k^n(s,t)) = \begin{cases} (k^n(s,2t), e_0), & 0 \leqslant t \leqslant \frac{1}{2} \\ (e_0, k^n(s,2t-1)), & \frac{1}{2} \leqslant t \leqslant 1; \end{cases}$$

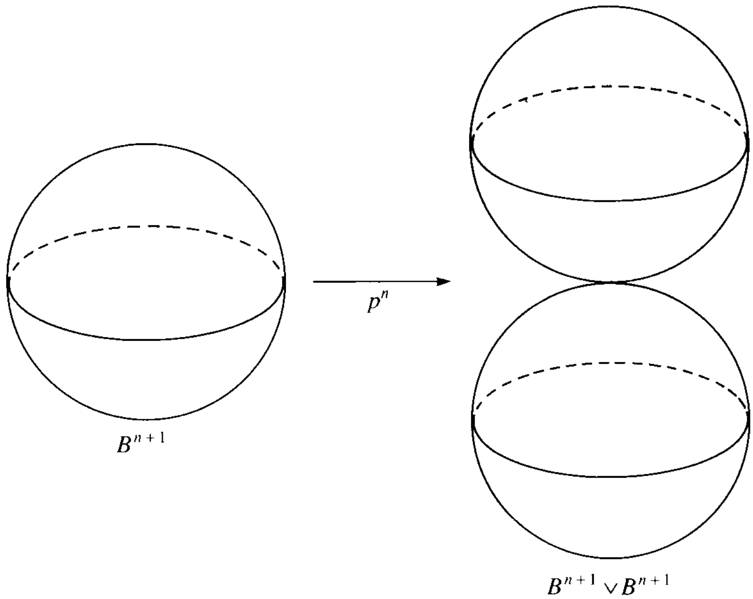

Figure 5

this means that the points with last coordinate equal to zero are mapped into the wedge point (e_0, e_0).

The maps p^n induce the *pinching of the spheres*

$$\dot{p}^n : S^n \to S^n \vee S^n.$$

(The symbol $\vee$ denotes the usual *wedge product*: for any pair of based spaces, say (X, x_0), (Y, y_0), the space $X \vee Y$ is defined to be $X \times \{y_0\} \cup \{x_0\} \times Y$, regarded as a subspace of $X \times Y$.)

An inaccurate but graphic description of the pinching is provided by cell division, a basic process in biology.

For $n \geqslant 2$, there is another useful type of *pinching*:

$$\hat{p} : B^{n+1} \to B^{n+1} \vee B^{n+1}$$

given by

$$\hat{p}^n(k^n(k^{n-1}(s, u), t)) = \begin{cases} (k^n(k^{n-1}(s, 2u), t), e_0), & 0 \leqslant u < \frac{1}{2}, \\ (e_0, k^n(k^{n-1}(s, 2u-1), t)), & \frac{1}{2} \leqslant u \leqslant 1. \end{cases}$$

This means that the points with penultimate coordinate equal to zero are mapped into the wedge point (e_0, e_0).

Next, consider the map obtained by projecting $B^{n+1} \times I$ onto $B^{n+1} \times \{0\} \cup S^n \times I$ from $(0, 2)$ in $\mathbf{R}^{n+1} \times \mathbf{R}$ (see Figure 6):

$$r^{n+1} : B^{n+1} \times I \to B^{n+1} \times \{0\} \cup S^n \times I$$

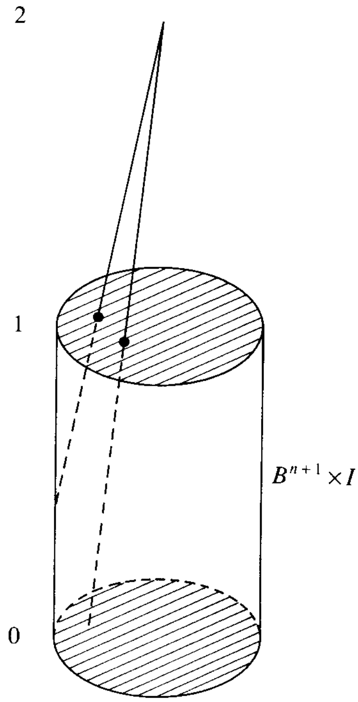

Figure 6

$$r^{n+1}(s,t)=\begin{cases}\dfrac{2}{2-t}(s,0), & 0\leqslant t\leqslant 2(1-|s|),\\[2ex] \dfrac{1}{|s|}(s,2|s|+t-2), & 2(1-|s|)\leqslant t\leqslant 1, |s|\neq 0.\end{cases}$$

Notice that the restriction of r^{n+1} to $B^{n+1}\times\{0\}\cup S^n\times I$ is the identity and that the composition of r^{n+1} with the inclusion of the latter space into $B^{n+1}\times I$ is homotopic rel. $B^{n+1}\times\{0\}\cup S^n\times I$ to the identity map, via the homotopy

$$R^{n+1}: B^{n+1}\times I\times I\to B^{n+1}\times I$$

given by

$$R^{n+1}(s,t,u)=u(s,t)+(1-u)r^{n+1}(s,t);$$

thus, $B^{n+1}\times\{0\}\cup S^n\times I$ is a strong deformation retract of $B^{n+1}\times I$. This means that the inclusion of S^n in B^{n+1} is a closed cofibration (see Example 1, Section A.4).

The restriction of the homotopy R^n to $B^n\times\{1\}\times I\simeq B^n\times I$ factors through the map h^n, thereby inducing a homeomorphism (see Figure 7)

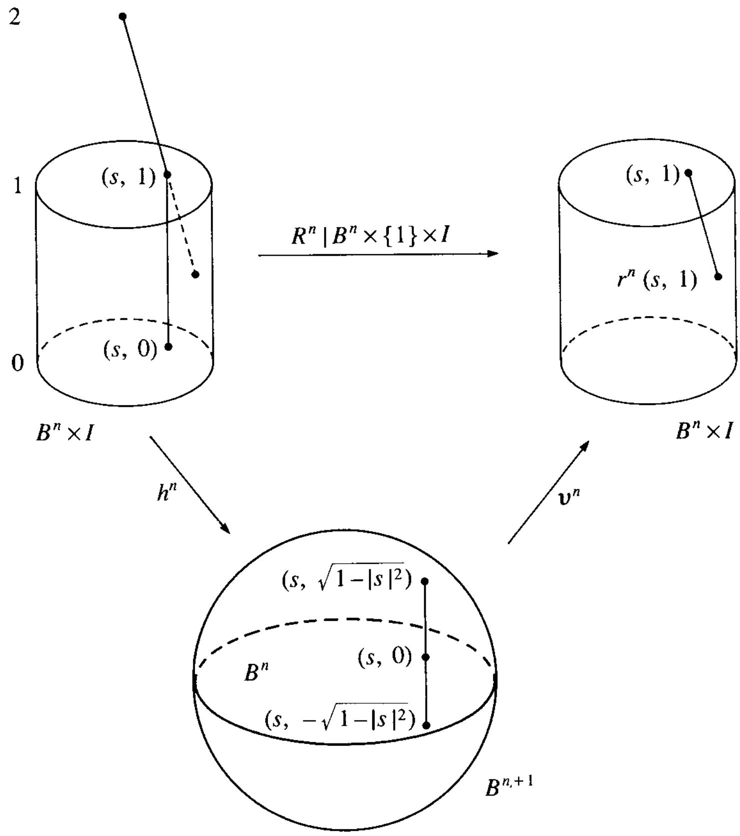

Figure 7

$$v^n : B^{n+1} \to B^n \times I;$$

one should notice that, regarding i_+, i_- as inclusions of B^n into B^{n+1}, then

$$v^n \circ i_- = r^n | B^n \times \{1\} \quad \text{and} \quad v^n \circ i_+ = \text{inclusion}.$$

The homeomorphism v^n, interesting in its own right, can be used to interchange the components $B^n \times \{0\} \cup S^{n-1} \times I$ and $B^n \times \{1\}$ of the boundary of $B^n \times I$: to see this, first note that v^n maps the upper hemisphere of S^{n+1} onto $B^n \times \{1\}$ and its lower hemisphere onto $B^n \times \{0\} \cup S^{n-1} \times I$; the actual interchange is then effected by the composite function $w^n = v^n \circ l^n \circ (v^n)^{-1}$. Two more remarks about the map v^n are called for: firstly, v^n induces a homeomorphism

$$\dot{v}^n : S^n \to B^n \times \dot{I} \cup S^{n-1} \times I;$$

secondly, v^n combines with the two pinchings p^n and $\hat{p}^n$ to yield an interesting commutative property:

Lemma 1.0.1 *For every $n \geqslant 1$,*

(i) *there is a unique map*

$$q^n : (B^n \vee B^n) \times I \to (B^n \times I) \vee (B^n \times I)$$

such that

$$q^n \circ (p^n \times 1) \circ v^n = (v^n \vee v^n) \circ \hat{p}^n;$$

(ii) *the map*

$$\hat{q}^n : (B^n \times I) \vee (B^n \times I) \to (B^n \vee B^n) \times I$$

induced by the obvious inclusions is a left homotopy inverse to q^n: there is a homotopy $\hat{q}^n \circ q^n \simeq 1$ rel.$((e_0, e_0), 1)$ and transforms the boundary of $(B^n \vee B^n) \times I$ into itself. □

In order to have enough fun in this game of balls and balloons, one actually needs more than one ball and one balloon in every dimension. Thus every space homeomorphic to the ball B^n (respectively, $\mathring{B}^n$) is called an *n-ball* (respectively, *open n-ball*) and every space homeomorphic to the sphere S^n is called an *n-sphere*. If B is any $(n+1)$-ball, its boundary sphere i.e., the image of S^n under a homeomorphism $B^{n+1} \to B$, is denoted by δB.

Proposition 1.0.2 *For any non-negative integers p and q, $B^p \times B^q$ is a $(p+q)$-ball with boundary sphere $B^p \times S^{q-1} \cup S^{p-1} \times B^q$; moreover, for every $n > 0$, $(B^1)^n$ is an n-ball.*

Proof Define $\Phi : B^p \times B^q \to B^{p+q}$ by setting, for every $(s, s') \in B^p \times B^q$,

$$\Phi(s, s') = \{\max(|s|, |s'|)/\sqrt{|s|^2 + |s'|^2}\}(s, s'),$$

if $(s, s') \neq (0, 0)$ and

$$\Phi(0, 0) = 0.$$

The continuity of Φ is not difficult to prove. Its inverse is obtained as follows. Let $s = (s_1, \ldots, s_p, \ldots, s_{p+q}) \in B^{p+q}$ be given. Set $s' = (s_1, \ldots, s_p)$ and $s'' = (s_{p+1}, \ldots, s_{p+q})$; then, define

$$\Phi^{-1}(s) = \{|s|/\max(|s'|, |s''|)\}(s', s'').$$

The restriction of Φ to $\delta(B^p \times B^q)$ gives the second homeomorphism announced in the statement. The third homeomorphism is obtained by induction on n. □

Projective spaces

From the topological point of view, projective spaces are intimately connected to spheres. However, before exhibiting this connection, one must give the definition of 'projective space' over a field.