Modules and rings

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Preface

In his or her journey through the book, how is the novice reader to identify the major results, which are like milestones marking one's progress along the way? First of all, some of them carry special names. Thus items named after a person are important, e.g. Wedderburn Theorems, Baer criterion, Jacobson's radical, Schanuel's Lemma, Fitting's Lemma, Krull-Remak-Schmidt Theorem, etc. To further help the reader, the paragraph headings "theorem", "proposition", and "construction" are used infrequently as opposed to the more numerous usage of "lemma", "corollary", "consequence", "observation", an unlabeled paragraph, and "remark". Here these terms are in descending order of importance, "theorem" being the most and "remark" the least important. In order to especially facilitate the actual identification of major results, the word "theorem" in this book is used more sparingly than in any comparable text known to the author. For example, with the exception of the preliminary review of basics in Chapter 0, the first time this word occurs is in Chapter 3 (in 3-3.3). In this particular case, it deserves the name "theorem" because it is a result that is difficult to prove, not at all obvious, but a result which once grasped is like an open passageway allowing free and easy access to the whole theory of injectivity. Also, it (Theorem 3-3.3) is an important and useful tool in more advanced module theory. Here in this text, also the word "proposition" marks significant mathematical statements and is used grudgingly. To summarize – in other texts the reader has to decide which theorems are important and which are not, but not in this book. There just are no unimportant theorems.

This text has more material than can be covered in a one year course. Although there are no real prerequisites aside from linear algebra, an introductory abstract algebra course, which usually includes groups, fields, and some commutative rings, might be helpful. Thus the book is intended for upper level undergraduates and beginning graduate students, as well as anyone else wishing to learn ring and module theory from the ground up.

Students taking a course prefer to follow a textbook without much skipping, omitting and jumping from chapter to chapter. For this reason the order in which the topics are presented here is the same in which they are usually taught in many introductory courses in many universities.

The first eight chapters with some possible omissions are designed for a one semester introductory course for advanced undergraduates or first year graduate students. This begins with free, projective, and injective modules, and tensor products of modules as well as algebras. Then free, tensor, and finite dimensional exterior algebras are discussed. The more abstract construction of arbitrary exterior algebras as quotients of tensor algebras could be omitted. At this point the course assumes a definite sense of direction: simple modules, primitive rings, the Jacobson radical, and subdirect products. In the second semester, from then on logically (with primitive replaced by prime and Jacobson radical replaced by prime radical), the next step would be to go on to primes, semiprimes, and the prime radical. Another possibility is to go directly to the Wedderburn Theorems instead. The above two semester course is not merely a theoretical possibility, but has been taught several times by the author, however, with the following optional material omitted: Appendix 5; 3-5 Noetherian Rings; 10-2 Projective Dimension; and possibly even omitting also 10-1.14 through 10-1.21. 6-6 More on Density and Simples; 7-3 Local Rings; 9-4 Nil Radicals; 9-4 Primes and Semiprimes in Derived Rings; alternate approaches to Wedderburn Theorems, 10-3 Direct Proofs; 10-4 Uniqueness; and 10-5 Rings with D.C.C. and Idempotents.

This book does not dwell too long on any one topic and thus is suitable for courses where a wide range of topics have to be covered quickly. This is also the reason why the chapters on category theory, functors, module categories, and more complicated facts about tensor products are at the end of the book.

One often hears from one's colleagues and students that in all too many books it is not possible to read and understand one topic only, without being forced to read the whole book and to decipher unnecessarily intertwined chapters and notation. Here in this book a deliberate attempt is made to make each chapter as self contained as is possible. In each chapter the main results are obtained as quickly, and as simply as possible. At the end of each chapter some more specialized and peripheral material which can be omitted is found. For example, sometimes several self contained and distinct proofs of the same theorem are given; thus providing the reader some real behind the scenes insight. In this way the book may also be of service to the student and working mathematician alike, who may wish to learn or review some less elementary topic quickly, e.g. category theory, simple algebras, or hereditary rings.

Although Abelian categories would suffice for applications to modules, nevertheless category theory is covered in greater generality so as to make it also applicable to other fields, such as topology or partially ordered systems, where greater generality is required.

In Chapters 6-12, the ring need not necessarily have an identity with the exception of the first two sections of Chapter 10 (i.e. 10-1 and 10-2). An identity is assumed in the sections on free, projective, and particularly injective modules. In these isolated cases, where an identity is not assumed once an adequate notation and terminology is established, it is just as easy to prove everything for a general ring. If all our theorems were proved for rings with an identity, they could not be applied to subrings of a ring when they arise naturally during the course of some proof. Some theorems should be tools capable of proving other theorems. And occasionally we point out that in some arguments an identity element is not needed. However, throughout almost the whole book, the ring is assumed to have an identity. Whenever a ring has an identity element, we always assume that all modules are unital. Finally, if the reader wishes, he or she can simply assume throughout that all rings have an identity.

This book also tries to teach how to do ring and module theory, rather than strive for encyclopedic coverage. There is a whole chapter which develops useful module construction techniques, like pullbacks, pushouts, and applies these to give an alternate quite different proof of the existence of the injective hull of a module. Simiarly, the chapter on module categories and the chapter on flat modules develop quickly some of the more important facts, and do not cover these subjects exhaustively.

The last chapter on systems of equations in modules, pure projectivity, pure injectivity, and pure injective hulls is somewhat more advanced. At this point in time, the author does not know of any textbook in print which develops this subject logically from the beginning as is done here.

As already stated some of the great theorems are mentioned in the introductions to the chapters and the chapters themselves. For the benefit of the newcomer to module and ring theory we will mention a few more here next. Just what is the definition of a "major result or theorem"? The author does not have an absolute universally valid definition. It may even vary from one person to the next, depending upon one's needs and goals in module and ring theory, and it may vary with the passage of time for the same person. The reader may find it an interesting and profitable exercise to list a half a dozen or so general criteria which characterize and identify major mathematical results. For these reasons the reader should not take any list, and in particular the one here, as the final absolute truth, but rather as a road map to guide him or her through the text. The author found one such criterion on a poster on the undergraduate mathematics bulletin board at M.I.T . "Beware of a theorem that counts something." It could possibly be more important than you think because in all likelyhood it can be used in novel ways and in circumstances that are beyond your imagination when you first read such a theorem. The first Wedderburn Theorem (7-1.39) counts something. It says that a simple ring R with the descending chain condition on right ideals is isomorphic $R \cong M_n(D)$ to an $n \times n$ matrix ring $M_n(D)$ over a division ring D. If in addition R happens to be a finite dimensional algebra over a commutative field F, then there are two intergers, not only n but also the dimension of D over F. To the future development of ring and module theory, the Wedderburn or Wedderburn-Artin Theorems were what Columbus was to the development of the Americas. A few of the original papers in which it appeared are the following: [Wedderburn 08], [Artin 27], [Noether 29], and [Hopkins 39]. For a possibly more complete list, see [Goodearl 76; p. 184].

Unlike other sciences, where the objects to be studied may be unequivocally given, in mathematics they have to be invented or discovered by means of lengthy constructions. Such constructions are particularly important, because they expand the reader's repertoire of mathematical objects thus giving a new framework within which to understand mathematics and physics. Such constructions enlarge one's mental categories or concepts. Thus the acquisition of a construction is like obtaining a new previously missing sense modality, such as touch, hearing, or sight. Many such constructions involve universal mapping properties, such as free modules (Chapter 1), tensor products of modules, tensor products of algebras, exterior algebras (Theorem 5-2.11, Theorem 5-3.19), pullbacks and pushouts. If one stops and examines the idea of a universal property it is really a surprising device frequently used in 20th century algebra. An object or set is determined or defined in terms of entities of a totally different nature—functions. Coming from this perspective for the beginning novice reader at least, the Universal Mapping Theorem (4-3.5) is noteworthy. It gives necessary and sufficient conditions on an algebra C to be the algebra tensor product of given algebras A and B. Here the algebra tensor product C of A and B is defined in terms of linear disjointness of A and B as subsets of C. Many other texts either omit tensor products of algebras altogether, or merely regard them as tensor products of vector spaces with additional properties.

The beginning student may find the construction of tensor and exterior algebras illuminating and useful in Chapter 5. Aside from finite matrix rings, these two are among the most frequently used algebras in areas other than algebra, and even in physics. In applications many students find that these two algebras are used without an adequate explanation of what they really are thus making the whole application to appear logically unsound. Here these two algebras are developed from two different perspectives. Also in various applications these algebras are used in different ways which are suited to either one or the other of our perspectives.

This is a book about modules just as much as about rings, and the construction and properties of the injective hull of a module (Theorem 3-4.13, 3-3.3) is one of the major results. The latter theorem immediately implies that any module can be embedded in an injective module, and that any given module is injective if and only if it is a direct summand of any module containing it as a submodule. It is this last property which explains the importance of injective modules in module and ring theory as a whole. Frequently, the presence of direct summands either simplifies or more accurately makes certain module and ring theoretic proofs even possible to begin with.

The dual of an injective module is a projective module, and there is a dual development of the theory also for projective modules. The proofs tend to be a little shorter and easier, and for this reason none of the results in the sections on projectives (10-1, 10-2) is labelled as a theorem. However, the main results here are the equivalent characterization of projective modules (Proposition 10-1.5) and the proof that any module has a well defined projective dimension (10-2.6).

The Jacobson Density Theorem (6-5.7) says that if any ring R has a faithful simple module V, that then V is a vector space over a certain division ring D and that R is almost equal to the full ring of D-linear transformations of this vector space. Even in the early 1990's new and more complicated versions of it are the subjects of research papers and talks. What makes this theorem (and others in general) interesting is that it takes something abstract and on the surface not very informative, like R being a primitive ring, and shows that this abstract condition is completely equivalent to R being something that is quite familiar, and well understood. Historically, it possibly may be one of the more influential single theorems, such as the Hilbert Nullstellensatz (9-4.10) and Wedderburn-Artin Theorem (7-1.39). In fact, the density theorem may be viewed as a successor of the Wedderburn Theorem.

Unquestionably, the whole theory of the Jacobson radical was one of the most influential developments in the evolution of ring theory (chapter 7). Here it is hard to single out a few theorems. It is the logical coherent development of the whole theory that counts. Here to single out the most important theorem would be like asking for the most important part of a painting by Monét.

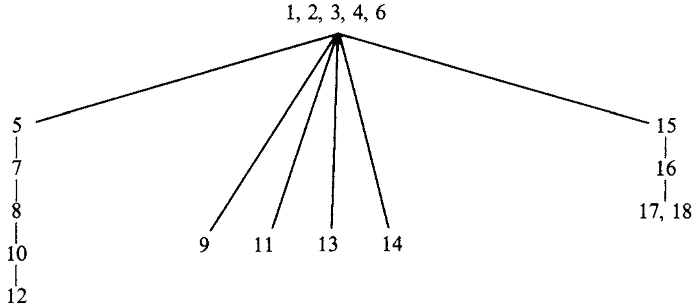
Theorem 9-2.8 characterizes the prime radical of an arbitrary ring, while Theorem 9-2.16 characterizes semiprime rings. These two theorems show that the rich and elaborate theory of primeness for commutative rings, surprisingly also has an analogous theory for noncommutative rings. During the early part of the twentieth century ring and module theorists tended to devote most of their efforts to rather specialized rings like finite dimensional algebras, commutative rings, or division algebras. It was the success of results such as the Jacobson radical and the theory of primeness developed in the context of an arbitrary associative ring which established noncommutative ring and module theory as a subject in its own right.

For a right Noetherian ring R, the prime radical is the sum of all the nilpotent right (or left) ideals of R (11-3.7) and by a theorem of Levitzki (11-3.6) is nilpotent. A right Artinian ring R is automatically right Noetherian (11-3.5). Let us ask for some kind of converse of this. Suppose that R is right Noetherian to start with. What additional hypotheses on R will force R to be also right Artinian? The answer is given by the Hopkins-Levitzki Theorem (11-3.8) which in some sense rounds out and complements the Wedderburn-Artin Theory by also looking at Noetherian rings. Noetherian rings are diverse and complicated, and in general not neatly describable. And the Hopkins-Levitzki Theorem gives us a clear picture of a natural class of right Noetherian rings.

The Krull-Schmidt Theorem (11-4.9) and a generalization (Theorem 11-4.8) gives conditions under which a module is a direct sum indecomposable modules which is unique up to isomorphism of the indecomposable summands. These theorems also describe certain interesting substitution properties, whereby some direct summands may be replaced by others. This whole theory culminates in the difficult to prove and significant Krull-Remak-Schmidt Theorem 11-4.11 which gives conditions when a module can be decomposed as a direct sum of indecomposable modules in an essentially unique way. The reader will find that the main results throughout are indicated by a syntactic (sparse) use of the word "theorem", and they are identified in the introductions to the chapters, and in the chapters themselves. This last part of the preface was for the benefit of novice student in order to guide that student well into the text.

Note to the reader

First a brief guide is given to the reader who may not wish to read the whole book but only certain select chapters. Roughly speaking, the dependance or independance of various chapters is summarized in the following diagram.



Moreover, in Chapter 1, section 1-5. Direct and Inverse Limits may be omitted, except for Chapters 16–18, where these are used. In later applications of Chapter 7, only 7-1.1 through 7-1.36 is used. Also Chapter 9 is almost independent from the other chapters, although some defintions and a few easy facts from Chapter 9 are occasionally used in Chapter 10.

At the expense of being overly repetitious, throughout, if the ring has an identity element, then always all modules are unital. If one wishes to take a middle ground between assuming that all rings have an identity, or that as a rule rings need not have an identity, then one should assume an identity element for all chapters – except 6 and 7.

There is some benefit in knowing some mathematical facts, without having the ability to prove these facts. For this reason, the reader should read the exercises even if he or she does not wish to do them.

Throughout an attempt has been made to display results and theorems by means of formulas and logical symbols, and to even incorporate logical symbols in English sentences where this seems to give more clarity and emphasis. The reader should expect to find symbols such as $\exists$, $\forall\Leftrightarrow$, and $\Rightarrow$ in some sentences, particularly in statements of lemmas and proposition, because the meaning of these symbols is always clear. Words can be hard to grasp and sometimes even ambiguous. Also, for the sake of brevity, some of the standard abbreviations of terminology are used, such as "map" for right R-module homomorphism, and "$G \to G/N$" as an abbreviation for the natural quotient homomorphism.

Many important topics are not covered in the book, particularly those which are a bit more advanced and well presented in other books. Thus if one wishes to pursue some topic further beyond this book, or to see it treated from another perspective, the bibliography is a good place to start. No attempt has been made to give credit for most theorems and proofs. The inclusion of an author in the bibliography should not be interpreted to mean that this author is the main contributor necessarily to this area. And certainly many major ring and module theorists and authors are not mentioned in the bibliography.

Two appendices are provided. Appendix 0 gives a review of some basic facts such as equivalence relations, Zorn's lemman, ordinals, cardinals, and the four isomorphism theorems. Appendix 5 to Chapter 5 gives additional information about tensor algebras and their quotients.

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