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CHAPTER 1

Modules

Introduction

The basic nomenclature for modules and module homomorphisms is defined. Direct sums and products of modules are introduced. Split short exact sequences are discussed. Existence and universal properties of direct and inverse limits are established.

Direct limits generalize direct sums, inverse limits – direct products. This topic is covered in Chapter 26-6, but could very well be covered at the end of the present Chapter 1. The construction of direct and inverse limits of modules and rings is a good exercise in using all the concepts introduced in this Chapter 1. Furthermore, they are a rich source of nontrivial examples of modules and rings.

1-1 Definitions

Throughout, R is an arbitrary ring with or without an identity element.

1-1.1 Definition. An additive abelian group M with addition denoted by $+$ is a right *R-module* if there is a function $M \times R \to M$, $(m, r) \to mr$, for $m \in M$, $r \in R$, such that for any $x, y \in M$ and any $a, b \in R$ the following hold:

(i) $(x + y)a = xa + yb$, $x(a + b) = xa + yb$;
(ii) $x(ab) = (xa)b$.

Notation. The notation $M = M_R$ will mean that R is a ring and M is a

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right R-module (and similarly $V = {}_RV$ for left modules). The zero module will be denoted by either one of the three $\{0\} = (0) = 0$.

For the remainder of this section, $M = M_R$.

1-1.2 Definition. $M = M_R$ is *unital* if (i) $1 \in R$, i.e. the ring has an identity element 1, and (ii) $x1 = x$ for all $x \in M$.

1-1.3 Definition. For any $M = M_R$, an additive subgroup $N \subseteq M$ is a right R-*submodule* if $nr \in N$ for all $n \in N$ and all $r \in R$. Submodules will be denoted by $N \leqslant M$ or $N < M$. The symbols $\leqslant$ and $<$ will be used exactly the same way as $\subseteq$ and $\subset$. I.e., if the case $N = M$ is trivial, we sometimes will write $N < M$; $\leqslant$ and $<$ are used for special emphasis, and whenever important, the fact that $N \neq M$ will be explicitly stated.

A submodule $N < M$ is *proper* if $N \neq M$. If $M \neq (0)$, then $(0) < M$ is proper.

Terminology. By a module will be meant a right module, by a submodule — a right submodule.

Remark. If R is commutative, the distinction between right and left disappears.

1-1.4. Definition. For rings E and R suppose that $M = M_R$ and also that $M = {}_EM$. Then M is left E, right R-*bimodule* if for any $\psi \in E$, any $r \in R$, and any $m \in M$, $(\psi m)r = \psi(mr)$.

Example. If $M = R$ and $E = R$, then R is a left R, right R-bimodule.

1-1.5 Definition. For any ring R, for any $M = M_R$ and $W = W_R$, an additive abelian group homomorphism $f: M \to W$ is a right R-module *homomorphism* (also called module map, or R-map, or R-homomorphism) if $f(mr) = (fm)r$ for all $m \in M$ and $r \in R$.

If f is one-to-one it is called a *monomorphism*, or monic; if f is onto — *epimorphism*, or epic. If f is both one-to-one and onto, then it is an *isomorphism* and denoted by $M \cong W$. The symbols "$\rightarrowtail$", "$\twoheadrightarrow$", and "$\rightarrowtail\!\!\!\!\twoheadrightarrow$" will occasionally be used to indicate monics, epics and isomorphisms. A homomorphism of a module M into itself is called an *endomorphism*.

The symbol $\mathrm{Hom}_R(M, W)$ denotes the additive abelian group of all right R-module homomorphisms $f, g: M \to W$, $(f + g)m = fm + gm$,

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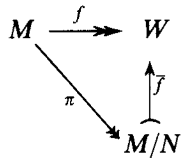
$m \in M$; its zero element is the zero homomorphism $0: M \to (0)$. If $W = M$, define $\text{End}_R M \equiv \text{Hom}_R(M, M)$. Then $\text{End}_R M$ is a ring under the above addition and composition; $f, g: M \to M$, $(fg)(m) \equiv f(gm)$. The identity endomorphism $1: M \to M$, $1m = m$, is the identity element of $\text{End}_R M$, and M is a unitial left $\text{End}_R M$ module. Furthermore, M is a left $\text{End}_R M$ and right R-bimodule.

1-1.6 Definition. For a right R-submodule $N \subseteq M$, let $\bar{M} = M/N$ be the quotient abelian group. Define $\bar{M} \times R \to \bar{M}$ by $(m + N)r = mr + N$, $m \in M, r \in R$. Then $\bar{M}$ is a right R-module called the quotient module. The R-module map $\pi: M \to \bar{M}$, $\pi m = m + N$, $m \in M$, is called the natural projection.

1-1.7 Let $f: M \to W$ be an R-module homomorphism, and let $N \leqslant M$ and $V \leqslant W$ be submodules. Then $fN = \{fn \mid n \in N\} \leqslant W$ and $f^{-1}V = \{m \in M \mid fm \in V\} \leqslant M$ are submodules. In particular, if $N = M$ then the *image* of f is $fM = \text{image} f = \text{im} f$; if $V = (0)$, then the *kernel* of f is the submodule $f^{-1}(0) = \text{kernel } f = \ker f = f^{-1}0$. The *cokernel* of f is coker $f = W/fM$.

1-1.8 Lemma. *Let $f: M \to W$ be an epic homomorphism of R-modules. Define $N = \text{kernel} f$. Let $\pi: M \to M/N$ be the natural R-module homomorphism, and define $\bar{f}: M/N \to W$ by setting $\bar{f}\bar{x} = fy$, where $\bar{x} \in M/N$ and $y \in \bar{x}$ is arbitrary. Then $\bar{f}$ is an R-module isomorphism, $f = \bar{f}\pi$, and $M/\ker f \cong fM = W$.*



1-1.9 Notation. For subsets $A \subset M = M_R$ and $B \subset R$, the set $AB \subset M$ is defined as $AB = \{\Sigma_{i=1}^{n} a_i b_i \mid n = 0, 1, \ldots; a_i \in A, b_i \in B\}$. By convention, when $n = 0$, the empty sum is $0 \in M$, and usually $AB \neq \{ab \mid a \in A, b \in B\} \neq AB \backslash \{0\}$. If $A = \{a\}$, write $\{a\}B \equiv aB$.

The next definition gives a large source of modules which will play an important role in later development.

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1-1.10 Definition. Regard $R = R_R$ as a right R-module. A *right ideal* is an abelian subgroup $B \subseteq R$ which is a right R-submodule of R. (For $R = {}_RR$, left ideals are defined analogously.)

An abelian subgroup $A \subset R$ is a right ideal if and only if $AR \subseteq A$.

Thus the notation for a right ideal is $B < R$ or $B \leqslant R$.

A set $I \subseteq R$ is an *ideal* if I is both a right and a left R-submodule of R, denoted by $I \lhd R$. The symbol "$\lhd$" denotes ideals of any ring whatever, where equality is allowed.

1-1.11 Annihilators. For any subset $Y \subseteq M = M_R$, define $Y^\perp \subseteq R$ by $Y^\perp = \{r \in R \mid \forall\, y \in Y,\ yr = 0\}$, i.e. $Y^\perp = \{r \mid Yr = \{0\}\}$. Then $Y^\perp$ is a right ideal of R called the *annihilator* of Y.

Three special cases will occur frequently:

(i) $m \in M$, $Y = \{m\}$; define $m^\perp = \{m\}^\perp = \{r \in R \mid mr = 0\}$. Left multiplication by m defines a right R-module homomorphism $R_R \twoheadrightarrow mR \subseteq M$, $r \to mr$, $r \in R$. Since the kernel is $m^\perp$, $mR \cong R/m^\perp$.

(ii) if $Y \subseteq M$ is a submodule, then $Y^\perp \lhd R$ is an ideal.

(iii) In particular, if $Y = M$, then $M^\perp \lhd R$ is an ideal. Define $\bar{R} = R/M^\perp$. Then M is also a right $\bar{R}$-module, where for $m \in M$ and $\bar{a} \in \bar{R}$, $m\bar{a} \equiv mb$ for any $b \in \bar{a}$. According to the next definition, $M_{\bar{R}}$ will be a faithful $\bar{R}$-module.

1-1.12 Definition. A module $V = V_R$ is *faithful* if for any $r \in R$, $Vr = (0)$ forces $r = 0$. Alternatively, V is faithful if and only if for any $0 \neq r \in R$, also $Vr \neq (0)$.

Remark. Let $\mathbb{Z} = 0, \pm 1, \pm 2, \ldots$. The endomorphism ring of the additive abelian group V is $\mathrm{End}_{\mathbb{Z}} V = \mathrm{Hom}_{\mathbb{Z}}(V, V) \supseteq \mathrm{Hom}_R(V, V)$. There is a ring homomorphism $R \to \mathrm{End}_{\mathbb{Z}} V$, where $r \in R$ gives the map $r: V \to V$, $v \to vr$ for $v \in V$. Thus V_R is faithful as an R-module if and only if this ring homomorphism is monic, in which case R can be viewed as a subring in $R \subseteq \mathrm{End}_{\mathbb{Z}} V$.

1-1.13 Suppose that M and W are given R-modules which are annihilated by an ideal $B \lhd R$, i.e. $MB = 0$ and $WB = 0$, and suppose that $f: M \to W$ is an R-module homomorphism. Then both M and W are right $\bar{R} = R/B$-modules in a natural way (as in 1-1.11(iii)), and furthermore, f is also an $\bar{R}$-homomorphism.

In particular, for any R-map $f: M \to W$ whatever, since $(fM)M^\perp = 0$,

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the corestriction $f: M \to fM$ of f to $fM \leqslant W$ is a homomorphism of $R/M^{\perp}$-modules M and fM.

1-1.14 Examples. Let $\mathbb{Z} = 0, \pm 1, \pm 2, \ldots$.

1. Any abelian group A is a $\mathbb{Z}$-module, where for $0 < n \in Z$, $a \in A$, $na = an = a + \cdots + a$, and $(-n)a = a(-n) = (-a) + \cdots + (-a)$, n-times.
2. Every additive abelian group is a left $\text{End}_{\mathbb{Z}} A$ module. For example, if elements of $A = \mathbb{Z} \times \mathbb{Z}$ are viewed as column vectors, then $\text{End}_{\mathbb{Z}} A$, which consists of 2×2 matrices over $\mathbb{Z}$, is noncommutative.
3. Let V be a finite dimensional vector space over a field F and $T: V \to V$, $v \to vT$; $v \in V$, a linear transformation. Then V is a right module over the ring $R = F[x]$ of polynomials over F, where for $v \in V$ and $p(x) \in F[x]$, $vp(x) = vp(T)$. The R-submodules of $W \leqslant V$ are precisely the T-invariant subspaces, i.e. ordinary vector subspaces W such that $WT \subseteq W$.

Let $m(W; x)$ be the minimal polynomial of the restriction $T \mid W$ of T to W, and $m(v; x)$ the minimal polynomial of v. then

(i) $v^{\perp} = Rm(v; x) \lhd R$;
(ii) $W^{\perp} = Rm(W; x) \lhd R$.

The ring $F[x]$ is a Euclidean domain, hence a principal ideal domain, which in turn implies that it is a unique factorization domain.

1-2 Direct products and sums

1-2.1 For an arbitrary family of modules M_i, $i \in I$, indexed by an arbitrary index set, the product $\Pi\{M_i \mid i \in I\} \equiv \Pi M_i$ is defined as the set of all functions $\alpha, \beta: I \to \cup\{M_i \mid i \in I\}$ such that $\alpha(i) \in M_i$ for all i, which becomes an R-module under pointwise operations, $(\alpha - \beta)(i) = \alpha(i) - \beta(i)$, and $(\alpha r)(i) = \alpha(i) r$ for $r \in R$.

(ii) Alternatively, the product can be viewed as consisting of all strings or sets

$$x = \{x_i \mid i \in I\} \equiv (x_i)_{i \in I} \equiv (x_i) \equiv (\underline{\quad\quad}, x_i, \underline{\quad\quad}), \; x_i \in M_i; \; i\text{-th}.$$

If $x = (x_i)$, $y = (y_i)$, and $r \in R$, then $x - y = (x_i - y_i)$, and $xr = (x_i r)$.

The direct sum $\oplus\{M_i \mid i \in I\} \equiv \oplus M_i$ is defined as the submodule $\oplus M_i \subseteq \Pi M_i$ consisting of all those elements $x = (x_i) \in \Pi M_i$ having at most a finite numbers of nonzero *coordinates* or *components* x_i. Some-

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times $\oplus M_i \subseteq \Pi M_i$ are called the external direct sum and external direct product.

In general, whenever the index set is suppressed in symbols like ΠM_i, $\oplus M_i$, $\cup M_i$, (or $\cap M_i$), it will be understood that i ranges over the largest possible index set, namely all of I, and never over some subset of I.

The R-module maps $j_k: M_k \to \Pi M_i$, $j_k(m) = (0 \_\_\_ 0, m, 0 \_\_\_ 0)$, where the k-th coordinate is $m \in M_k$, and $\pi_k: \Pi M_i \to M_k$, $\pi_k[(x_i)_{i\in I}] = x_k$ are called the inclusion and projection maps. Throughout, $\delta_{ik} = 1$ if $i = k$ and 0 if $i \neq k$. Thus $\pi_k j_k = 1: M_k \to M_k$ is the identity map, and $\pi_i j_k = \delta_{ik}$.

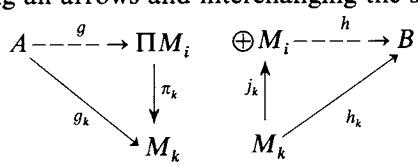
If $\Pi\{N_i \mid i \in I\}$ is another direct product of modules N_i over the same index set I, and $f_i: M_i \to N_i$ are any R-maps, then $\Pi\{f_i \mid i \in I\} = \Pi f_i: \Pi M_i \to \Pi N_i$ is the R-map defined by $(\Pi f_i)(x_i)_{i\in I} = (f_i x_i)_{i\in I}$. Since $\Pi f_i(\oplus M_i) \subseteq \oplus N_i$, define $\oplus\{f_i \mid i \in I\} = \oplus f_i$ to be the restriction and corestriction of Πf_i to the direct sums, i.e. $\oplus f_i: \oplus M_i \to \oplus N_i$. In case all $N_i = N$ are the same module the following notations are frequently used: $\Pi N_i = \Pi\{N \mid I\} = N^I$, and $\oplus N_i = \oplus\{N \mid I\} = N^{(I)}$. Define an R-map $s: \oplus\{N \mid I\} \to N$ by $s[(n_i)_{i\in I}] = \Sigma\{n_i \mid i \in I\} \in N$, where all $n_i \in N$. In this special case, the composite of $\oplus f_i$ followed by s is the map $s \circ (\oplus f_i): \oplus M_i \to N$, $s \circ (\oplus f_i)(x_i)_{i\in I} = \Sigma\{f_i x_i \mid i \in I\}$, where $x_i \in M_i \subset \oplus M_i$.

Note that $\oplus M_i = \Pi M_i$ if and only if $|I| < \infty$, the index set is finite.

1-2.2 Universal properties of products and sums. Let $\oplus M_i \subseteq \Pi M_i$, j_k, and π_k be as before. (i) For any module A and any set of R-homomorphisms $g_k: A \to M_k$, for all $k \in I$, there exists a unique R-homomorphism $g: A \to \Pi M_i$ such that $\pi_k g = g_k$ for all $k \in I$.

(ii) For any module B and R-maps $h_k: M_k \to B$, $k \in I$, there exists a unique R-map $h: \oplus M_i \to B$ such that $h j_k = h_k$ for every $k \in I$.

Note that (i) and (ii) say that there are commutative diagrams and that there is a "duality" – either diagram can be obtained from the other by reversing all arrows and interchanging the sum with the product below.



Note also that for $a \in A$, $ga = \{g_i a \mid i \in I\} = (g_i a)$, and $h[(x_i)] = \Sigma\{h_i x_i \mid i \in I\}$ is a well defined element of B because the sum has only a finite number of nonzero terms if $(x_i) \in \oplus M_i$.

1-2.3 Remark. The previous universal properties uniquely determine

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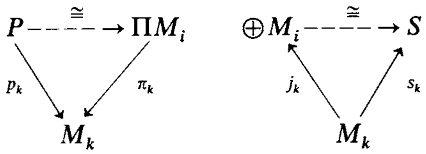
up to isomorphism, not only the direct product and sums $\oplus M_i \subseteq \Pi M_i$ as modules, but also the maps j_k and π_k.

Suppose that P and S are modules, and $p_k: P \to M_k$ and $s_k: M_k \to S$ maps, and that P and S have the exact same universal properties as the product ΠM_i and sum $\oplus M_i$ respectively. Then there are commutative diagrams where the horizontal maps are isomorphisms.



The two frequently used symbols "$\langle\ \rangle$", "Σ", and internal direct sum are next defined.

1-2.4 Definition. For a module M, and for any subset $T \subseteq M$, the *R-submodule generated* by T is denoted by $\langle T \rangle$, and is defined as $\langle T \rangle = \cap\{N \mid T \subseteq N,\ N \leqslant M\}$. If $T = \{x\}$ is a singleton, abbreviate $\langle\{x\}\rangle = \langle x \rangle$.

1-2.5 If I is any index set and $M_i \subset M$, $i \in I$, an indexed set or family of submodules, then define $\Sigma\{M_i \mid i \in I\} = \langle \cup\{M_i \mid i \in I\}\rangle$. This will frequently be abbreviated as $\Sigma M_i = \langle \cup M_i \rangle$, where whenever the index set is suppressed, it will always be assumed that i ranges over the largest possible index set, that is all of I, and never over some subset of I. Thus

$$\sum_{i \in I} M_i = \left\{ x_1 + \cdots + x_n \,\middle|\, \begin{array}{ll} \forall\, 0 \leqslant n \in \mathbb{Z}; & i(1), \ldots, i(n) \in I; \\ \forall\, x_k \in M_{i(k)}, & k = 1, \ldots, n \end{array} \right\}$$

1-2.6 The sum ΣM_i is an *internal direct sum* if for any $j \in I$, $M_j \cap \langle \cup\{M_i \mid i \in I,\ i \neq j\}\rangle = (0)$, written as $\Sigma M_i = \oplus\{M_i \mid i \in I\} = \oplus M_i$. Since it will be amply clear from the context whether an internal or external direct sum is meant, the two will be denoted by the same symbol.

More explicitly, the above sum by definition is direct if for any $1 \leqslant k$, $n \in \mathbb{Z}$, and for any n-distinct unequal indices $i(1), \ldots, i(n) \in I$, $M_{i(k)} \cap [M_{i(1)} + \cdots + M_{i(k-1)} + M_{i(k+1)} + \cdots + M_{i(n)}] = (0)$ for all $k = 1, \ldots, n$.

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1-2.7 Modular law. Let A, B, and C be submodules of a module. Then

(i) $A \subseteq C \subseteq A + B \Rightarrow C \cap (A + B) = A + (C \cap B)$;
(ii) $A \subseteq C \subseteq A \oplus B \Rightarrow C \cap (A \oplus B) = A \oplus (C \cap B)$.

1-2.8 Observations. For $T \subseteq M$ and $\mathbb{Z} = 0, \pm 1, \pm 2, \ldots$ the following hold.

1. $\langle T \rangle$ is the unique smallest submodule of M containing T.
2. $\langle M \rangle = M$, $\langle M \backslash \{0\} \rangle = M$.
3. For $x \in M$, $\langle x \rangle = \{nx + xr \mid n \in \mathbb{Z}, r \in R\} = \mathbb{Z}_x + xR$. If $M = R_R$, then $\langle x \rangle$ is the principal right ideal generated by x.
4. $\langle T \rangle = \Sigma\{\mathbb{Z}T + tR \mid t \in T\}$; if $1 \in R$, $\langle T \rangle = \Sigma\{tR \mid t \in T\}$.
5. If $T = \varnothing$, then $\langle \varnothing \rangle = (0)$. In this case when $T = \varnothing$, the empty sum by definition is equal to $0 \in M$. (Later the empty intersection will have to be interpreted as the whole space.)

1-2.9 Direct sum conditions. Let $M = M_R$, let I be an arbitrary index set, and $M_i \subset M$, $i \in I$, right R-submodules. The following are all equivalent.

(i) ΣM_i is an internal direct sum.
(ii) For any $1 \leqslant n$ and any choice of $n + 1$ distinct indices $i(0)$, $i(1), \ldots, i(n) \in I$, $M_{i(0)} \cap [M_{i(1)} + \cdots + M_{i(n)}] = (0)$.
(iii) Each $0 \neq m \in M$ has a unique representation $m = m_{i(1)} + \cdots + m_{i(n)}$, all $0 \neq m_{i(k)} \in M_{i(k)}$.
(iv) For any $1 \leqslant n$, and any distinct $i(1), \ldots, i(n) \in I$, if $m_{i(k)} \in M_{i(k)}$ and $m_{i(1)} + \cdots + m_{i(n)} = 0$, then all $m_{i(k)} = 0$.

1-2.10 Corollary 1. *Assume in addition that I is well ordered by "$\leqslant$". Then the above conditions* (i)–(iv) *are equivalent to* (v).

(v) For any $2 \leqslant n$, and any choice of $i(1) < i(2) < \cdots < i(n) \in I$, $M_{i(n)} \cap [M_{i(1)} + \cdots + M_{i(n-1)}] = (0)$.

1-2.11 Corollary 2. *If $I = \{1, \ldots, n\}$ is finite, then* (i)–(iv) *are equivalent to* (v)

(v) for any $2 \leqslant j \leqslant n$, $M_j \cap [M_1 + \cdots + M_{j-1}] = (0)$.

1-3 Adjunction of 1 to R

Throughout this section $\mathbb{Z}$ is the ring $\mathbb{Z} = 0, \pm 1, \pm 2, \ldots$ of integers.

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1-3.1 For any ring R whatever $\mathbb{Z} \times R$ becomes a ring under

$$(n, a) + (m, b) = (n + m, a + b).$$
$$(n, a)(m, b) = (nm, nb + ma + ab) \quad n, m \in \mathbb{Z};\ a, b \in R.$$

The identity element of $\mathbb{Z} \times R$ is $e \equiv (1, 0)$ irrespective of whether R has one or not. If one identifies $R \equiv \{0\} \times R \lhd \mathbb{Z} \times R$, $r \equiv (0, r)$, $r \in R$, then the above become

$$(ne + a) + (me + b) = (n + m)e + (a + b),$$
$$(ne + a)(me + b) = nme + nb + ma + ab.$$

1-3.2 Let R be any ring M and N any R-modules, and $f: M \to N$ an R-homomorphism. Then M (and N) becomes a unital $\mathbb{Z} \times R$-module under the definition $x(n, a) = x(ne + a) = nx + xa$, $x \in M$, $(n, a) \in \mathbb{Z} \times R$. Furthermore, f is also a $\mathbb{Z} \times R$-homomorphism. The set of R-submodules of M is exactly the set of $\mathbb{Z} \times R$-submodules of M.

1-3.3 **Definition.** For any ring R, define R^1 to be the ring

$$R^1 = \begin{cases} R & \text{if } 1 \in R; \\ \mathbb{Z} \times R & \text{if } 1 \notin R. \end{cases}$$

1-3.4 **Remarks.** (1) For any $M = M_R$, and any $x \in M$, $\langle x \rangle = xR^1$. (2) For any $x \in R$, the smallest ideal containing x is $\mathbb{Z}x + xR + Rx + RxR = R^1 x R^1$.

1-3.5 **Observation.** Let $1 \in R$ and $M = M_R$. Define $M^0 = \{m \in M \mid m1 = 0\}$ and $M^1 = \{m \in M \mid m1 = m\}$. Then

(i) $M^0 = \{m - m1 \mid m \in M\}$ and $M^1 = \{m1 \mid m \in M\}$ are submodules;
(ii) $M = M^0 \oplus M^1$;
(iii) $M^0 R = (0)$ while M^1 is unital;
(iv) UNIQUE: If $m = P \oplus Q$ with $PR = 0$ and Q unital, then $P = M^0$ and $Q = M^1$.

Proof. Conclusions (i) and (iii) are easy as well as that $M^0 \cap M^1 = (0)$. (ii) For $m \in M$, $m = (m - m1) + m1$ and (i) show that $M = M^0 \oplus M^1$. (iv) Since $P1 \subseteq PR = (0)$, $P \subseteq M^0$. Hence $M^1 = M1 = (P \oplus Q)1 = P1 + Q1 = Q1$ and $Q = Q1 = M^1$. By the modular law, $M^0 = M^0 \cap (P \oplus Q) = P + M^0 \cap Q = P$ because $M^0 \cap Q = M^0 \cap M^1 = 0$.

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1-3.6 Important convention. From now on it is always assumed that if the ring R has an identity, then all modules are unital.

1-4 Sequences of modules

1-4.1 Definition. Let $\{M_i \mid i \in I\}$ be a set of modules together with module maps $f_i: M_i \to M_{i-1}$ indexed by a finite or infinite convex subset $I \subseteq \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Then the sequence

$$\ldots M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \longrightarrow \ldots$$

is *exact* if kernel f_i = image f_{i+1} for all i except possibly the smallest and largest one. In sequences, abbreviate (0) = 0.

Remarks. Let $f: M \to W$ be an R-map.

1. By omitting the 0 and pushing "$0 \to M$" and "$W \to 0$" through the M and W and shortening the arrows the following notations for monics and epics have arisen:

   f is monic $\Leftrightarrow 0 \to M \to W$ is exact ($M \rightarrowtail\!\!\twoheadrightarrow M$),
   f is epic $\Leftrightarrow M \to W \to 0$ is exact ($M \twoheadrightarrow W$).

2. For any f, the following is exact

   $0 \to \ker f \to M \to \operatorname{im} f \to \operatorname{coker} f.$

1-4.2 Definition. An exact sequence with five modules of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is called a *short exact* sequence.

Consequences. 1. $A \cong \operatorname{im} \alpha$. 2. $C \cong B/\ker \beta$. 3. by use of 1 and 2, frequently a general short exact sequence of the above form may be replaced by $A \subset B \twoheadrightarrow C = B/A$.

1-4.3 Definition. An exact sequence $0 \to A \xrightarrow{\alpha} B$ *splits* if there exists an R-map $p: B \to A$ such that $p\alpha = 1_A$, the identity on A.

Similarly, an exact sequence $B \xrightarrow{\beta} C \to 0$ *splits* if there exists $q: C \to B$ such that $\beta q = 1_C$, the identity on C.

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