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(early winter 1691–2)

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PART 2

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(*c.* 1693)

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PART 3

CARTESIAN ANALYSIS OF HIGHER PLANE CURVES AND FINITE-DIFFERENCE APPROXIMATIONS

(*c.* summer 1695)

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§1. Simple schemes of subtabulation by finite differences. [1] (Add. 3965.18: 671ʳ/671ᵛ). *Ad hoc* rules for bisection of an interval by supposing a quadratic interpolating parabola, 673. And a cubic one, 674. [2] (Add. 3965.18: 697ʳ). Briggsian rules for quinquisection derived from first principles by supposing a cubic approximating parabola, 675. [3] (Add. 3965.18: 695ᵛ). The first five orders of differences of the ordinates, given at unit-intervals of the abscissa, of a quintic approximating parabola, 677. Applied to derive (not quite correctly) a more accurate rule for quinquisecting terms given at unit-intervals of the argument, 678. The deduction is left off unfinished, 679.

§2. Determining the slope of the apparent path of the comet of 1680/1 at its sighting (by Flamsteed) on 30 December. [1] (Add. 3965.14: 586ʳ). First, by fitting a quartic parabola to the path: Newton's observational data as derived from Flamsteed, 682. The first four orders of divided differences of the latitudes with respect to longitude as base: the ensuing parabola (not quite accurately computed) is tested to fit the sighting on 13 January well enough, 684. Its slope at an arbitrary point (derived as the ratio of the fluxion of latitude to that of longitude), 684. Applied to compute the required slope on 30 December, 686. [2] (Add. 3965.11: 164ʳ). The angles made with the meridian on 30 December by the four great circles through the comet's sighted position in the celestial sphere on that day and also (one each) through its four other observed places, 686. The hyperbolic curve fitted to go through the Cartesian points which have the longitudes for base and the angles of slope to the meridian for ordinates, 688. Whence Newton computes the angle of intersection of the meridian with the great circle 'grazing' the comet's apparent path on 30 December; which is the slope to the meridian of the latter on that day, 688.

§3 (Add. 3964.4: 21r). 'Of Quadrature by (equidistant) Ordinates'. Case 1: the trapezoidal rule where there are just two ordinates, 690. Case 2: three equidistant ordinates, 690. The accurate 'Simpson' rule of approximation derived by adjusting 'errors' in twice applying the simple trapezoidal rule, 692. Case 3: the 'three-eighths' rule analogously derived where there are four ordinates, 692. Case 4: the Cotesian approximation where there are five ordinates accurately derived by considering similar 'errors' in making partial applications of Simpson's rule, 694. Newton's formula stated in Case 5 (where there are seven equidistant ordinates) is readily shown to be derived by analogously comparing the 'errors' ensuing from partial applications of the three-eighths rule, but errs in his choice of the wrong ratio of comparison; that in Case 6 (of nine ordinates) is accurately computed from Case 4, but here there is no longer a simple comparison factor as Newton assumes, 694. 'These are quadratures of the parabola passing through the end-points of the ordinates', 696. Supposing the general form of the approximation known and thence computing the coefficients from the simplest particular instances: where (Case 3) there are four equidistant ordinates, 696. The parallel calculation where (Case 4) there are five ordinates is incomplete in the surviving manuscript (and left so by Newton?), 698.

APPENDIX. Preliminary castings. [1] (Add. 3965.14: 611r). Case 5 (where there are seven equidistant ordinates) is inadequately approached by comparing 'errors' in partially applying the three-eighths rule, 700. A sequence of increasingly closer approximations to a curvilinear area on this basis, 701. [2] (*ibid.*: 612v). Case 4 (of five ordinates) is derived in an inferior form which, in effect, presupposes only a cubic (and not, as in accurate revise, a quartic) fitted parabola, 702.

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