PART I

SPECIAL CURVES

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Excerpt
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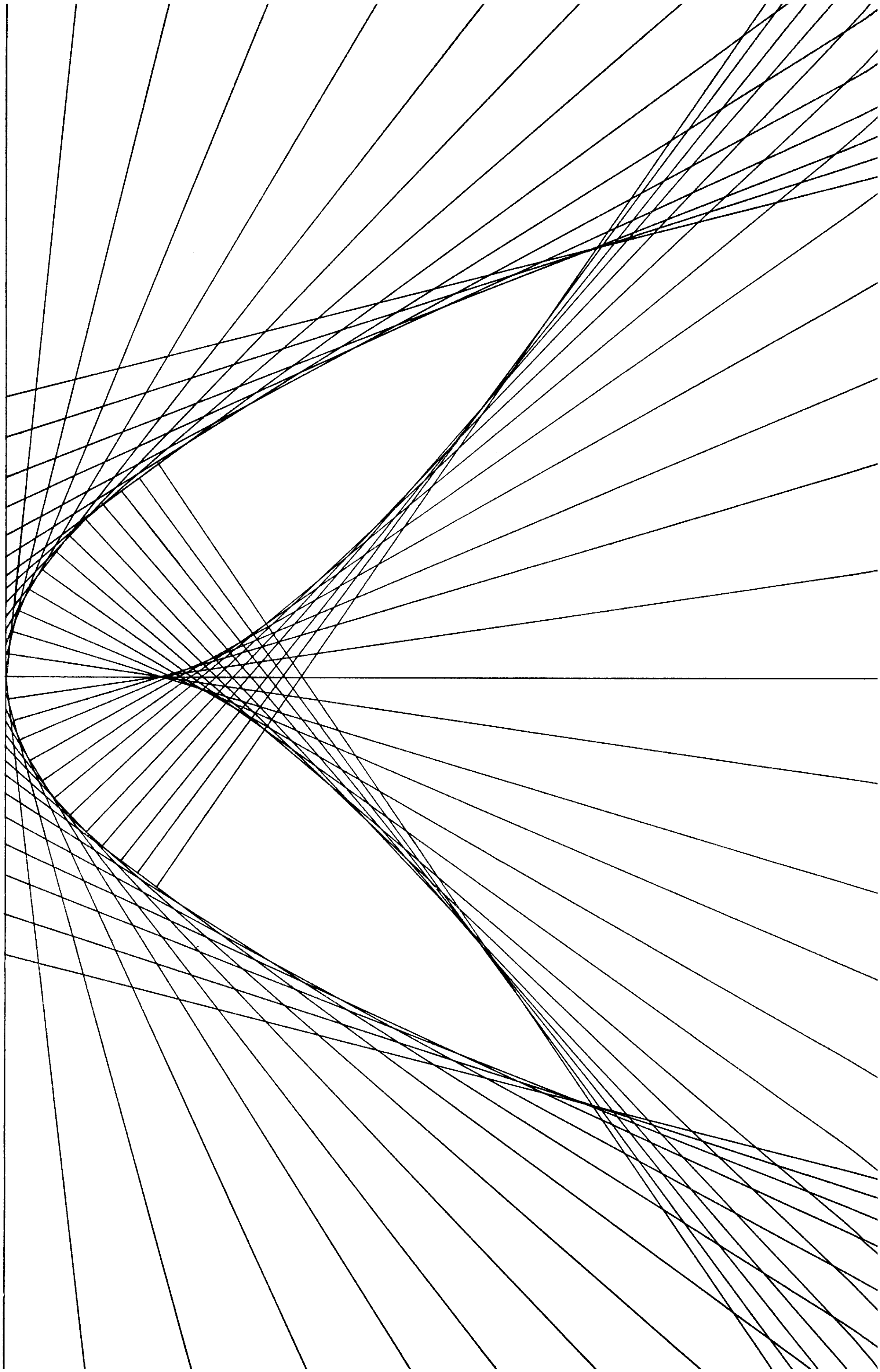

1

THE PARABOLA

To Draw a Parabola

Draw a fixed line *AY* and mark a fixed point *S*. Place a set square *UQV* (right-angled at *Q*) with the vertex *Q* on *AY* and the side *QU* passing through *S* (Fig. 2). Draw the line *QV*. When this has been done in a large number of positions, the parabola can be drawn freehand, touching each of the lines so drawn. The curve is said to be the *envelope* of the variable line *QV* (Fig. 1).

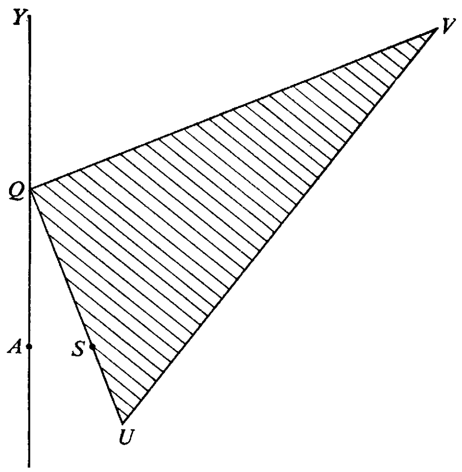

Fig. 2

Suitable Dimensions

With *AY* near to and parallel to the left-hand edge of the paper, the distance of *S* from *AY* should be approximately as follows:

Paper:	1_P	0·8 in. or	2 cm.
	2_P	1 in.	3 cm.
	3_P	1·5 in.	4 cm.

A second curve may be drawn on the same paper with the distance halved.

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Fig. 1. The parabola and its evolute

Geometrical Properties

In Fig. 3, SA is drawn perpendicular to AY. The curve is symmetrical about the axis AS and A is called the *vertex*. QP, qp are two positions of the variable line

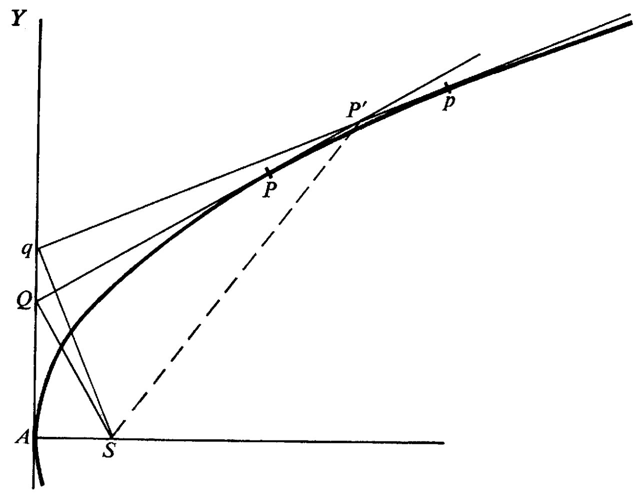

Fig. 3

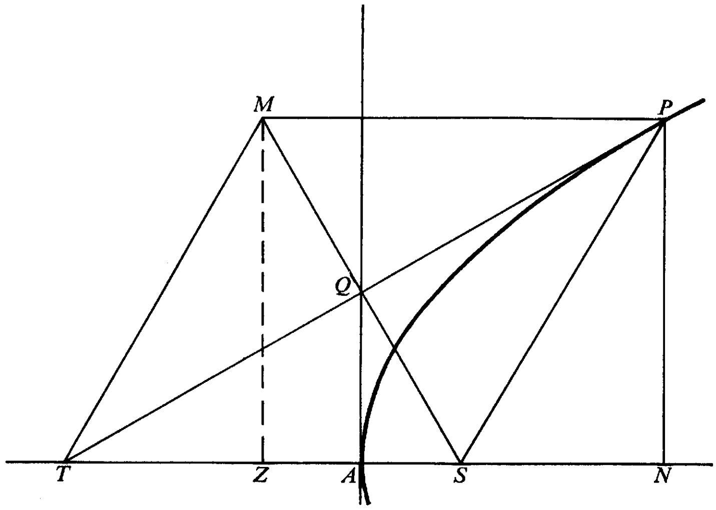

Fig. 4

(i.e. two tangents to the curve), intersecting at P'. SP' is joined. Then the points S, Q, q, P' are concyclic and angle AQS = angle $qP'S$. P' is not itself a point on the curve, but, the nearer together the two tangents are, the nearer to the curve will it be. Now imagine that q moves closer to Q. P' will move towards P and angle $qP'S$ will become angle QPS. Thus, in the limit, angle AQS = angle QPS. The point P is shown again in Fig. 4.

Focus and Directrix Property

It is seen from Fig. 4 that triangles SAQ, SQP are similar; hence

$$\text{angle } ASQ = \text{angle } QSP.$$

If PQ is produced to meet the axis at T, triangles SQT, SQP are congruent. Therefore $SP = ST$. If the rhombus $PSTM$ is completed, and MZ is drawn perpendicular to ST, then $SQ = QM$ and $SA = AZ$. It follows that Z is a fixed

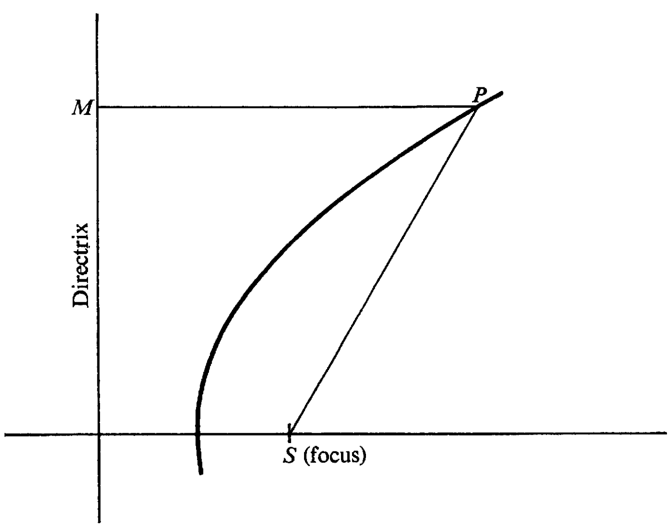

Fig. 5

point and MZ a fixed line. Moreover $SP = PM$. The parabola can thus be defined as the locus of a point P whose distance from a fixed point S (the *focus*) is equal to its distance PM from a fixed line (the *directrix*). This is shown in Fig. 5.

Cartesian Equation of the Parabola

If PN is the perpendicular from P to the axis (Fig. 4), $PN = 2QA$.

$$\therefore \quad PN^2 = 4QA^2 = 4AS.AT = 4AS.AN.$$

If AS and AQ are chosen as axes of coordinates of x and y respectively, and if P is the point (x,y), and $AS = a$, then $y^2 = 4ax$. This is the equation of the parabola.

Polar Equation of the Parabola

If, in Fig. 4, $SA = a$, $SP = r$ and angle $NSP = \theta$, then $SP = MP = ZN = ZS+SN$. Therefore $r = 2a+r\cos\theta$, and $r(1-\cos\theta) = 2a$, or $2a/r = 1-\cos\theta$. This is the polar equation of the parabola, referred to S as pole and SN as initial line. The equation $2a/r = 1+\cos\theta$ gives the same curve, turned through two right angles, since $\cos(180°+\theta) = -\cos\theta$.

Further Properties

The following may be proved as exercises:

1. If MP is produced to M', SP and PM' make equal angles with the curve (i.e. with the tangent to the curve at P).

This is the reflecting property of the parabola. If a mirror is made in the form of a paraboloid (i.e. the surface formed by rotating a parabola about its axis), rays from the focus S would be reflected into rays parallel to the axis. A searchlight beam is produced in this way. For the same reason rays coming in, parallel to the axis, would be focused at S. This is the way in which a reflecting telescope produces an accurate image of a star, free from spherical or chromatic aberration.

2. If PG (the *normal*) is drawn through P at right angles to the tangent PT, meeting the axis at G, $NG = 2a$ and is therefore constant. This gives a convenient method for drawing the normal at any point of the curve.

3. If PT cuts MZ at Y, PSY is a right angle. If PS is produced to meet the curve again at R, the tangent at R passes through Y. Angle SPY is equal to half angle RST, and angle SRY is equal to half angle PST; hence RYP is a right angle. Thus tangents at the ends of a *focal chord* meet at right angles on the directrix.

** 4. If a number of parallel chords of a parabola are drawn, their mid-points lie on a straight line parallel to the axis. This line is called a *diameter* of the parabola. The tangent at the point where it meets the curve is parallel to the chords and the tangents at the ends of any one of the chords meet on the diameter produced.

Hint: Let PP' be one of the chords and let PM and $P'M'$ be the perpendiculars from P and P' to the directrix. If K is the mid-point of MM', KS will be the radical axis of the two circles whose centres are P and P' and whose radii are PS and $P'S$. KS will thus be at right angles to the parallel chords and K will be a fixed point.

5. The two tangents from any point to a parabola subtend equal angles at the focus.

(*Hint:* If the tangents at P and P' meet at R, cutting the tangent at the vertex at Q and Q' respectively, the points S, Q, R, Q' are concyclic. Hence angle SRQ' = angle SQQ' = angle SPQ.)

Further Drawing Exercises

1. Draw any two lines and mark on each a series of points at equal intervals. (The intervals on the second line need not be equal to those on the first.) Call the points on the first line A_1, A_2, A_3, etc., and those on the second line B_1, B_2, B_3, etc. Join A_1B_1, A_2B_2, A_3B_3, etc. The envelope of these lines will be a parabola.

2. Use the above method (i) to draw a parabola to touch four given lines; (ii) to draw a parabola to touch two given lines at given points.

(*Hints:* (i) Take two of the given lines as the two fixed lines; let the other given lines cut them at A_1B_1 and A_nB_n. (ii) Take the two given lines as the two fixed lines; take the given points as A_1 and B_n, the intersection of the given lines being B_1 and also A_n.)

3. Draw normals at a large number of points on a parabola. (Use the property $NG = 2a$. It is convenient to mark the distance $2a$ along one edge of a set square, measuring from the right-angled corner.) The envelope of the normals drawn to any curve is called the *evolute* of that curve. The evolute of the parabola is a curve called the *semi-cubical parabola* (Fig. 1). It will be seen that from any point inside the evolute three normals can be drawn to the parabola, but from any point outside it only one.

4. Draw a circle cutting a parabola in four points and verify that the chords joining them in pairs are equally inclined to the axis.

5. Draw a circle cutting a parabola at the vertex and three other points. Verify that the normals at these three points are concurrent.

6. Given a parabola and two normals, use the last two results to draw a third normal concurrent with the first two.

7. Verify that the circumcircle of the triangle formed by three tangents to a parabola passes through the focus; and that the orthocentre of the same triangle lies on the directrix. (These properties are related to the Simson Line properties of the triangle.)

8. Draw several parabolas with the same vertex and axis, varying the position of S. Then draw a number of lines radiating from the vertex. This will illustrate the fact that all parabolas are geometrically similar: they have the same shape and vary only in size.

9. The semi-cubical parabola may be drawn as a locus as follows: Draw a fixed line AB and mark a fixed point O, not on the line. From O draw any pair of lines OL and OM, at right angles to each other, OL cutting the fixed line at Q. From Q draw QR perpendicular to AB, cutting OM at R. From R draw RP perpendicular to QR, cutting LO produced at P. Then P is a point of the locus.

Suitable dimensions. The fixed line should be near and parallel to a long edge of the paper. O may be about 2 in., or 6 cm., away from it. If graph paper is used points can be plotted very quickly by placing a ruler to represent OL and a set square with two of its sides representing OL and OM.

The Parabola: Summary

**Many of the following properties can be conveniently proved by the methods of the calculus and coordinate geometry, starting from one or other of the equations listed.

1. The Cartesian equation (origin at vertex) is $y^2 = 4ax$.

2. All parabolas are similar in shape, the constant a determining the size.

3. The polar equation (pole at focus) is $l/r = 1-\cos\theta$, where $l = 2a$.

4. The pedal equation is $p^2 = ar$.

5. Parametric equations are $x = at^2$, $y = 2at$.

6. $\psi = 180°-\phi, = \frac{1}{2}\theta, = \cot^{-1}t$.

7. $\rho = -2a(t^2+1)^{\frac{3}{2}}, = -(y^2+4a^2)^{\frac{3}{2}}/4a^2$.

8. The centre of curvature is $(2a+3at^2, -2at^3)$, and the evolute is the semi-cubic parabola $27ay^2 = 4(x-2a)^3$.

9. The area bounded by the curve, the x-axis and the ordinate is $\frac{2}{3}xy$ (i.e. two-thirds of the rectangle having the same base and height).

10. $s = a[t\sqrt{(1+t^2)}+\log_e\{t+\sqrt{(1+t^2)}\}]$.

11. The parabola is the section of a right circular cone by a plane making with the axis of the cone an angle equal to the semi-vertical angle.

12. It is the negative pedal of a straight line.

13. It is the locus of a point in a plane whose distance from a fixed point (the focus) is equal to its distance from a fixed line (the directrix).

14. It is the form assumed by a hanging chain under a uniform horizontal distribution of load (cf. the catenary, p. 119).

In nos. 15–20, the notation of Fig. 4 is used. S is the focus and ZM the directrix. $SA = AZ = a$. Let PT cut MZ at R and let PG be the normal at P, meeting the axis at G.

15. SQP is a right angle.

16. PQ bisects angle SPM.

17. $SP = PM = ST = SG$.

18. $NG = 2a$.

19. PSR is a right angle.

20. If PSP' is a focal chord, the tangents at P and P' meet at right angles on the directrix, at R.

21. From a given point outside the curve two tangents can be drawn and they subtend equal angles at the focus.

22. Three tangents to a parabola form a triangle whose orthocentre lies on the directrix and whose circumcircle passes through the focus.

23. From a given point inside the evolute three normals can be drawn to the

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parabola and their feet lie on a circle through the vertex; but from a point outside the evolute only one normal can be drawn.

24. The mid-points of a system of parallel chords lie on a straight line parallel to the axis. This line is called a *diameter*. The tangent at the point where it meets the curve is parallel to the chords; and the tangents at the ends of any one of the chords meet on the diameter produced.

The parabola was first studied by the Greeks as one of the sections of a cone. The earliest writer to show knowledge of these conic sections was Menaechmus (fourth century B.C.), a pupil of Plato and Eudoxus. He solved the problem of the duplication of the cube by drawing two parabolas (or, alternatively, a parabola and a hyperbola). This problem, to find the side of a cube which would have

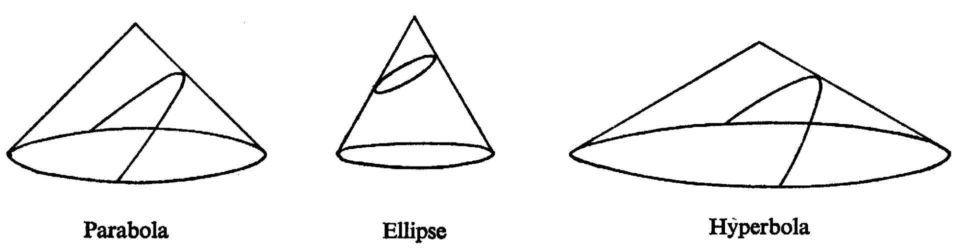

Fig. 6

double the volume of a given cube (in our notation, to solve the equation $x^3 = 2$ by a geometrical method), had been reduced to that of finding two geometric means between two given quantities, i.e. given a and b, find x and y such that $a/x = x/y = y/b$. The problem is insoluble by ruler-and-compass constructions, but Menaechmus solved it by finding the intersection of the parabolas $x^2 = ay$ and $y^2 = bx$. So Menaechmus evidently had some knowledge of these curves. He called the parabola a 'section of a right-angled cone', the ellipse a 'section of an acute-angled cone' and the hyperbola a 'section of an obtuse-angled cone'. This indicates that he had obtained the three curves as sections of three different right circular cones, the section being always at right angles to a generating line of the cone (Fig. 6).

Euclid wrote four books on conic sections, but they have been lost, perhaps because they were quickly superseded by the work of Apollonius (third century B.C.), 'the great geometer'. It was Apollonius who named the three curves; moreover he obtained all three from the same cone, by taking sections at different inclinations.

The origin of the names is as follows: In Fig. 7, V is the mid-point of a chord

QQ' and PV is a *diameter* (i.e. a straight line through the mid-points of a system of parallel chords). Apollonius drew a straight line PL, in the plane of the section, at right angles to PV, its length depending on the position of the section in relation to the cone. He then proved that, for the parabola, $QV^2 = PL.PV$; equivalent, in modern notation, to $y^2 = 4ax$, where x and y are oblique coordinates. The name *parabola* signifies 'equality', 'an exact comparison'.† The corresponding properties for the other two curves were $y^2 = 4ax - px^2$ and $y^2 = 4ax + px^2$; so he called one the *ellipse* ('falling short') and the other the *hyperbola* ('throwing beyond'). These names may be compared with the corresponding literary terms, *parable*, *ellipsis* and *hyperbole*.

Apollonius did not give the focus-directrix properties of the curves. These were first treated by Pappus of Alexandria (about A.D. 300).

The history of the conic sections begins again in the seventeenth century. The invention of coordinate geometry by Descartes put them in an altogether new light as curves of the second degree. His work on them, however, was incomplete and deliberately obscure. Wallis was the first to treat them systematically in this manner 'considered as plain Figures, exempted out of the Cone'.‡

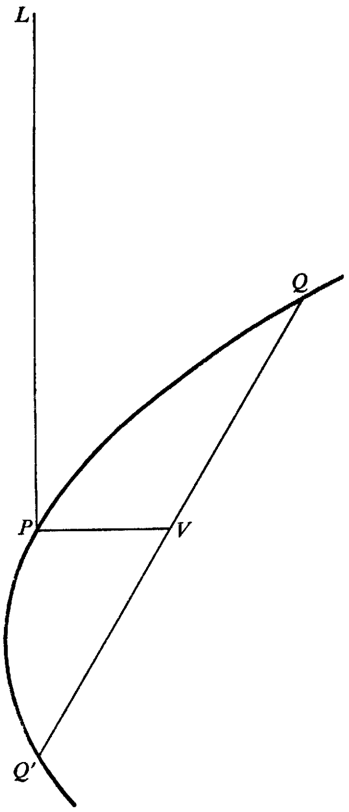

Fig. 7

A few years earlier, the young Pascal had treated them as projections of the circle, foreshadowing the projective geometry which was to develop 200 years later. About the same time, too, Galileo showed that the path of a projectile thrown obliquely was parabolic, a fundamental result in the science of ballistics.

The reflecting telescope was suggested by James Gregory in 1663 and the first one was made by Newton in 1668. Parallel rays are brought to a focus by a parabolic mirror, the focal length in large telescopes being anything up to 40 ft. The paraboloid form is also used in reflectors for searchlights and for radar receivers.

† 'Equality' as shown by 'application'; παραβάλλω had long ago taken on the derived meaning of *comparo*.

‡ So described in *Phil. Trans. R. Soc.* (1695). Wallis's work was published in 1655.