

Cambridge University Press

978-0-521-03287-2 - Ultrametric Calculus: An Introduction to p-Adic Analysis

W. H. Schikhof

Frontmatter

[More information](#)

---

CAMBRIDGE STUDIES IN  
ADVANCED MATHEMATICS 4

EDITORIAL BOARD

D.J.H. GARLING D. GORENSTEIN T.TOM DIECK P. WALTERS

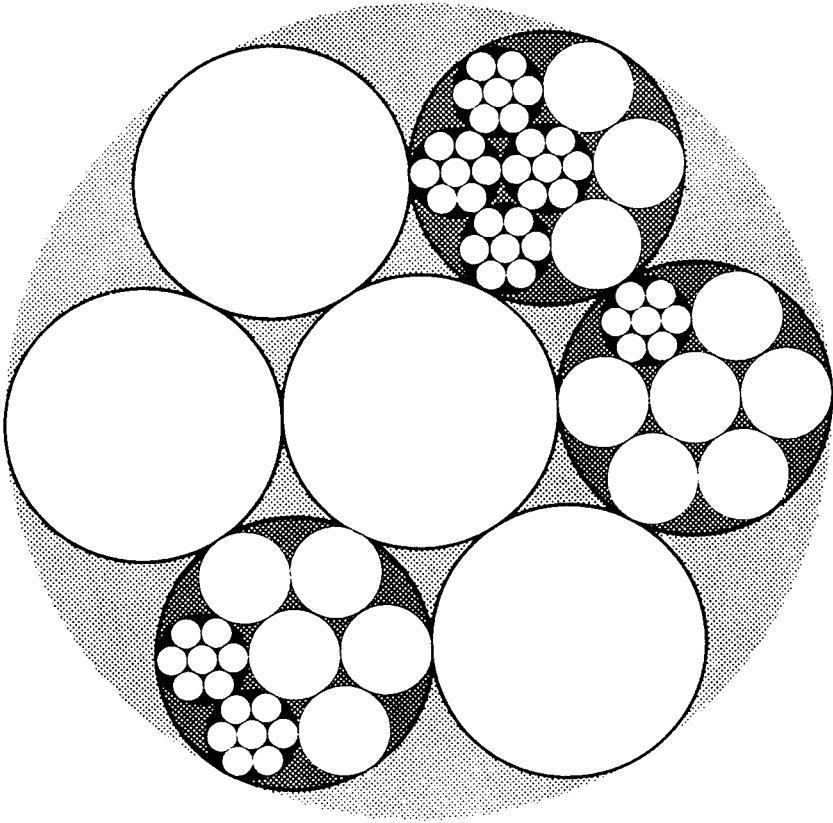
*Ultrametric calculus*

Cambridge University Press

978-0-521-03287-2 - Ultrametric Calculus: An Introduction to p-Adic Analysis

W. H. Schikhof

Frontmatter

[More information](#) $\mathbb{Z}_7$

Cambridge University Press

978-0-521-03287-2 - Ultrametric Calculus: An Introduction to  $p$ -Adic Analysis

W. H. Schikhof

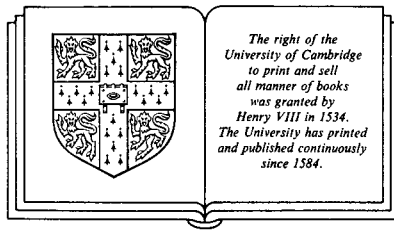
Frontmatter

[More information](#)

# Ultrametric calculus

An introduction to  $p$ -adic analysis

W. H. SCHIKHOF



CAMBRIDGE UNIVERSITY PRESS

Cambridge

London New York New Rochelle

Melbourne Sydney

Cambridge University Press  
978-0-521-03287-2 - Ultrametric Calculus: An Introduction to p-Adic Analysis  
W. H. Schikhof  
Frontmatter  
[More information](#)

---

CAMBRIDGE UNIVERSITY PRESS  
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press  
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9780521242349](http://www.cambridge.org/9780521242349)

© Cambridge University Press 1984

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without  
the written permission of Cambridge University Press.

First published 1984  
This digitally printed first paperback version 2006

*A catalogue record for this publication is available from the British Library*

*Library of Congress Catalogue Card Number: 83-7466*

ISBN-13 978-0-521-24234-9 hardback  
ISBN-10 0-521-24234-7 hardback

ISBN-13 978-0-521-03287-2 paperback  
ISBN-10 0-521-03287-3 paperback

# Contents

|  |    |
|--|----|
| <i>Frontispiece</i>                                    |    |
| <i>Preface</i>   | ix |
| <b>1 VALUATIONS</b>                                    | 1  |
| <b>Part 1: Valuations</b>                              | 1  |
| 1 Valuations   | 1  |
| 2 The strong triangle inequality                       | 2  |
| 3 The $p$ -adic integers                               | 4  |
| 4 The $p$ -adic numbers                                | 10 |
| 5 Topological properties of $\mathbb{Q}_p$             | 12 |
| 6 $\mathbb{Q}_p$ as a completion of $\mathbb{Q}$       | 14 |
| 7 $\mathbb{Q}_p$ compared to $\mathbb{R}$              | 16 |
| 8 Archimedean and non-archimedean valuations           | 18 |
| 9 Equivalence of valuations                            | 20 |
| 10 All valuations on $\mathbb{Q}$                      | 22 |
| 11 The residue class field and the value group         | 24 |
| 12 Series expansions of elements of $K$                | 27 |
| 13 Normed spaces                                       | 30 |
| 14 Extensions of valuations                            | 34 |
| 15 Uniqueness of the extended valuation                | 39 |
| 16 The valuation on the algebraic closure              | 42 |
| 17 Completion of the algebraic closure. $\mathbb{C}_p$ | 45 |
| <b>Part 2: Ultrametries</b>                            | 46 |
| 18 Ultrametric spaces                                  | 46 |
| 19 Compactness and separability                        | 51 |
| 20 Spherical completeness                              | 52 |
| 21 Best approximation                                  | 55 |
| <b>2 CALCULUS</b>                                      | 56 |
| <b>Part 1: Elementary calculus</b>                     | 56 |
| 22 The classical concepts of calculus                  | 56 |

| vi | <i>Contents</i>  |     |
|----|--|-----|
| 23 | Sequences and series   | 61  |
| 24 | Order-like structure in $K$                                      | 65  |
| 25 | (Locally) analytic functions                                     | 67  |
| 26 | Continuity and differentiability                                 | 73  |
| 27 | Continuously differentiable functions                            | 76  |
| 28 | Twice continuously differentiable functions                      | 81  |
| 29 | $C^n$ -functions   | 86  |
| 30 | Antiderivation and integration                                   | 93  |
|    | <b>Part 2: Interpolation</b>                                     | 98  |
| 31 | The idea of interpolation  | 98  |
| 32 | $p$ -adic exponents  | 100 |
| 33 | Roots of unity in $\mathbb{C}_p$ . The Teichmüller character     | 102 |
| 34 | $\sum_{n=0}^x a_n$ for a $p$ -adic integer $x$                   | 105 |
| 35 | The $p$ -adic gamma function                                     | 107 |
| 36 | A $p$ -adic Euler constant                                       | 110 |
| 37 | Values of $\Gamma_p$ in $\frac{1}{2}, 0, -1, -2, \dots$          | 111 |
| 38 | The $p$ -adic Gauss-Legendre multiplication formula              | 113 |
| 39 | Some other formulas involving $\Gamma_p$                         | 115 |
|    | <b>Part 3: Analytic functions</b>                                | 117 |
| 40 | Convergence of power series                                      | 117 |
| 41 | Substitution of power series                                     | 119 |
| 42 | The maximum principle  | 121 |
| 43 | Failure of the maximum principle for locally compact $K$ .       | 125 |
| 44 | $\exp$ and $\log$  | 128 |
| 45 | Extensions of $\exp$ and $\log$                                  | 129 |
| 46 | Trigonometric functions  | 135 |
| 47 | $(1+x)^a$  | 138 |
| 48 | The Artin-Hasse exponential                                      | 142 |
| 49 | $\arcsin$ and $\arccos$  | 143 |
|    | <b>3 FUNCTIONS ON <math>\mathbb{Z}_p</math></b>                  | 145 |
|    | <b>Part 1: Mahler's base and <math>p</math>-adic integration</b> | 145 |
| 50 | Orthogonal bases in Banach spaces                                | 145 |
| 51 | The Mahler base of $C(\mathbb{Z}_p \rightarrow K)$ .             | 149 |
| 52 | The Mahler coefficients. Examples                                | 152 |
| 53 | Mahler's base for $C^1(\mathbb{Z}_p \rightarrow K)$              | 158 |
| 54 | Mahler coefficients of $C^n$ -functions                          | 163 |
| 55 | The Volkenborn integral  | 167 |
| 56 | The Bernoulli numbers  | 171 |
| 57 | Integration over subsets   | 174 |

| <i>Contents</i>  | vii |
|--|-----|
| <b>Part 2: The <math>p</math>-adic gamma and zeta functions</b>    | 176 |
| 58 Local analyticity of $\Gamma_p$                                 | 176 |
| 59 A formula for $\log_p 2$  | 179 |
| 60 Diamond's log gamma function                                    | 182 |
| 61 The $p$ -adic zeta functions                                    | 185 |
| <b>Part 3: van der Put's base and antiderivation</b>               | 189 |
| 62 van der Put's base of $C(\mathbb{Z}_p \rightarrow K)$ .         | 189 |
| 63 Characterizations by means of coefficients                      | 193 |
| 64 Antiderivation  | 196 |
| 65 The differential equation $y' = F(x, y)$                        | 199 |
| 66 $C^1$ -solutions of a meromorphic differential equation         | 200 |
| 67 $p$ -adic Liouville numbers                                     | 203 |
| 68 van der Put's base of $C^1(\mathbb{Z}_p \rightarrow K)$         | 205 |
| <b>4 MORE GENERAL THEORY OF FUNCTIONS</b>                          | 208 |
| <b>Part 1: Continuity and differentiability</b>                    | 208 |
| 69 Convergent sequences of differentiable functions.               | 208 |
| 70 A function of the first class has an antiderivative             | 212 |
| 71 Points at which a differentiable function is $C^1$              | 215 |
| 72 Local behaviour of differentiable functions                     | 218 |
| 73 Lusin-type theorems   | 221 |
| 74 Differentiable homeomorphisms                                   | 224 |
| 75 Isometries  | 227 |
| 76 Extension theorems  | 230 |
| <b>Part 2: <math>C^n</math>-theory</b>                             | 234 |
| 77 Local invertibility of $C^n$ -functions                         | 235 |
| 78 Differentiation $C^n \rightarrow C^{n-1}$                       | 238 |
| 79 Antiderivation $C \rightarrow C^1$                              | 241 |
| 80 Antiderivation $C^{n-1} \rightarrow C^n$ . A candidate          | 243 |
| 81 Surjectivity of differentiation $C^n \rightarrow C^{n-1}$       | 247 |
| 82 Surjectivity of differentiation $C^\infty \rightarrow C^\infty$ | 249 |
| 83 $C^3$ -functions  | 252 |
| 84 Functions of two variables                                      | 255 |
| <b>Part 3: Monotone functions</b>                                  | 257 |
| 85 Sides of 0 in $K$   | 258 |
| 86 Monotone functions of type $\sigma$                             | 259 |
| 87 Monotonicity without type                                       | 262 |
| <b>APPENDIXES</b>  | 268 |
| <b>Appendix A Aspects of functional analysis</b>                   | 269 |

Cambridge University Press

978-0-521-03287-2 - Ultrametric Calculus: An Introduction to p-Adic Analysis

W. H. Schikhof

Frontmatter

[More information](#)

viii

*Contents*

|       |   |     |
|-------|---|-----|
| A. 1  | Two theorems on metric spaces                               | 269 |
| A. 2  | Functions of the first class of Baire                       | 271 |
| A. 3  | Orthonormal bases of $C(X \rightarrow K)$                   | 272 |
| A. 4  | The ultrametric Stone-Weierstrass theorem                   | 273 |
| A. 5  | Integration on compact spaces                               | 274 |
| A. 6  | Measures and distributions on $\mathbb{Z}_p$                | 281 |
| A. 7  | A substitution formula for real valued integrals            | 286 |
| A. 8  | The ultrametric Hahn-Banach theorem                         | 288 |
| A. 9  | A field with prescribed residue class field and value group | 288 |
| A. 10 | Isometrical embedding of an ultrametric space into $K$      | 293 |
|       | <b>Appendix B Glossary of terms</b>                         | 295 |
| B. 1  | Sets  | 295 |
| B. 2  | Subsets of $\mathbb{R}$                                     | 296 |
| B. 3  | Metric and topology   | 296 |
| B. 4  | Algebra   | 297 |
|       | <i>Further reading</i>                                      | 301 |
|       | <i>Notation</i>   | 302 |
|       | <i>Index</i>  | 304 |



Cambridge University Press

978-0-521-03287-2 - Ultrametric Calculus: An Introduction to  $p$ -Adic Analysis

W. H. Schikhof

Frontmatter

[More information](#)

## Preface

This elementary book is intended for advanced undergraduates or anyone on a higher level who wants to learn the basic facts of  $p$ -adic analysis. We only assume the reader to have some standard knowledge of analysis and algebra.

In analysis (and outside it) the fields  $\mathbb{R}$  and  $\mathbb{C}$  play a central role. For several reasons people started to study the implications of replacing  $\mathbb{R}$  or  $\mathbb{C}$  by a more general object, viz. a field  $K$  with a complete valuation  $|\cdot|$  comparable to the absolute value function (see Definition 1.1). Many such fields other than  $\mathbb{R}$  or  $\mathbb{C}$  exist, their valuations are all ‘non-archimedean’, i.e. they satisfy the ‘strong triangle inequality’  $|x + y| \leq \max(|x|, |y|)$ . The analysis in and over non-archimedean valued fields  $K$  is known as ultrametric (non-archimedean,  $p$ -adic) analysis.

In this book we shall treat the basic facts of ultrametric analysis together to form an alternative ‘one variable calculus course’. Thus, in  $K$  we shall consider familiar concepts such as continuity, differentiability, (power) series, integration, etc. However, the strong triangle inequality causes fascinating deviations from the ‘classical analysis’ (over  $\mathbb{R}$  or  $\mathbb{C}$ ); let us mention a few of them.

- (i) A series  $\sum a_n$  in  $K$  converges if  $\lim_{n \rightarrow \infty} a_n = 0$ . The power series  $\sum x^n/n!$  of  $\exp$  (if it makes sense at all) converges only on a disc strictly contained in the closed unit disc  $\{x : |x| \leq 1\}$ . Hence  $\sum 1/n!$  diverges (but  $\sum n!$  converges in many  $K$ ).
- (ii)  $K$  is not ordered. Yet it is possible to define ‘square roots of positive elements’ in a natural way. It may happen that  $\sqrt{16} = 4$  but  $\sqrt{25} = -5$ .
- (iii)  $K$  is not connected. In fact, each disc is open and closed; each point of a disc is a centre;  $\{x : |x| = 1\}$  is not the boundary of  $\{x : |x| < 1\}$ .
- (iv) Liouville’s theorem (a bounded analytic function  $K \rightarrow K$  is constant) holds if and only if  $K$  is *not* locally compact.
- (v) For each pair of continuous functions  $f_1, f_2 : K^2 \rightarrow K$  there exists an  $F : K^2 \rightarrow K$  such that  $\partial F/\partial x = f_1, \partial F/\partial y = f_2$ .

Cambridge University Press

978-0-521-03287-2 - Ultrametric Calculus: An Introduction to  $p$ -Adic Analysis

W. H. Schikhof

Frontmatter

[More information](#)*Preface*

x

During the past few decades ultrametric analysis has grown from a relatively small and remote area maintained by a few enthusiastic pioneers to a widely recognized and mature discipline. However, since there are so many parts of mathematics for which a corresponding ultrametric theory exists or is at least conceivable, the term ‘discipline’ is somewhat misleading. In fact, in the 1980 classification scheme of the *Mathematical Reviews* explicit references to  $p$ -adic and non-archimedean theories are headed under Number theory; Algebraic number theory, field theory and polynomials; Algebraic geometry; Group theory and generalizations; Topological groups and Lie groups; Real functions; Functions of a complex variable; Several complex variables and analytic spaces and Functional analysis. It is clear that in an elementary book like this we cannot go into all these different branches, each one of which has its own origin, motivation, problems, language and applications. On the contrary, this book is written deliberately from no specific background at all but takes a ‘naive’ approach.

Depending on his or her point of view the beginner may use this book (at least the first three chapters of it) either as a ‘main entrance’ to the various specialistic theories or as an introduction to  $p$ -adic analysis in its own right. The specialist should not look for deep theorems. Yet I hope that he or she will appreciate having the elementary facts together with the proofs collected into a single volume.

For the reader’s convenience many exercises of varying degree of difficulty have been included. They are meant for training purposes and also to indicate interesting by-paths. Exercises leading to results used in the main theory are marked\*. In Appendix A we discuss some themes that have something to do with the subject but do not fit into the main text, mostly because of their functional analytic character. Appendix B contains several basic definitions and facts needed in the book and is meant as a refresher course, only to be consulted if the reader needs part of it.

A small list of books is given for further reading; in some of these one may find extensive lists of references. Perhaps closest to our book is Mahler (1980), whereas Bachman (1964) is also elementary but moves towards algebraic theory. In Amice (1975) one can find more about analytic functions. Monna (1970) and van Rooij (1978) treat functional analysis. Iwasawa (1972) considers special functions of interest for number theory. Koblitz (1977), after an elementary start, moves towards the  $p$ -adic zeta functions and Dwork’s theory. More advanced studies and applications in algebraic number theory and algebraic geometry can be found in Koblitz (1980).

Only occasionally – in cases where a theorem is generally known as such – names of inventors are given. I have also added names to results that have not

Cambridge University Press

978-0-521-03287-2 - Ultrametric Calculus: An Introduction to p-Adic Analysis

W. H. Schikhof

Frontmatter

[More information](#)

*Preface*

xi

been published before and were pointed out to me by informal communication.

I am grateful to Lucien van Hamme and Arnoud van Rooij for their helpful comments and the stimulating effect they had on me.

October 1982

Wim Schikhof  
*Nijmegen, The Netherlands*