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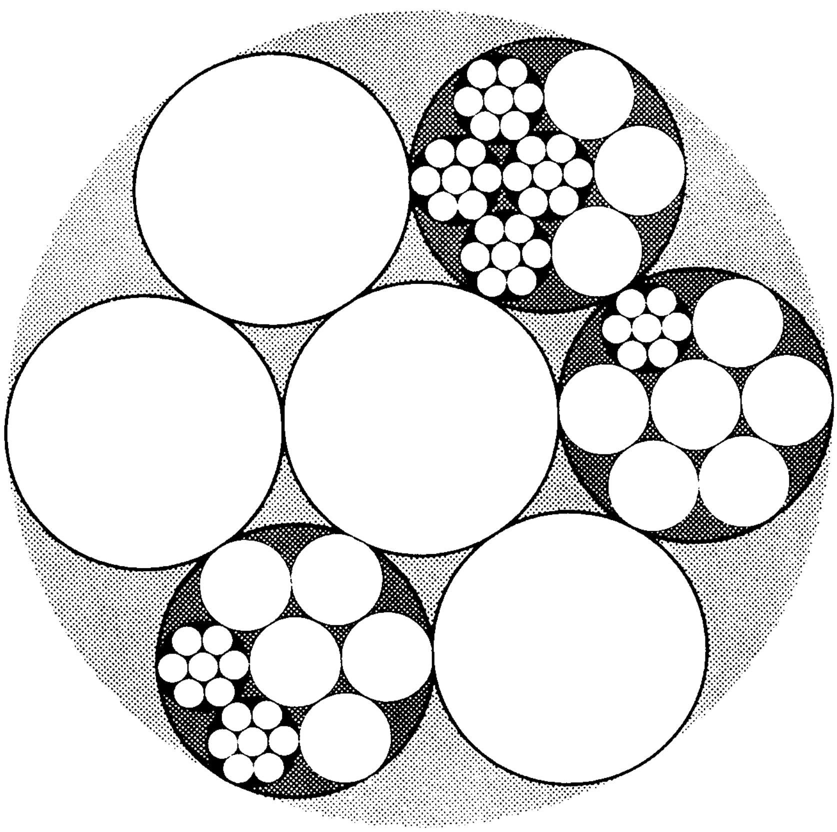
Ultrametric calculus



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Cambridge University Press
978-0-521-03287-2 - Ultrametric Calculus: An Introduction to p-Adic Analysis
W. H. Schikhof
Frontmatter
More information

Ultrametric calculus

An introduction to *p*-adic analysis

W. H. SCHIKHOF

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Contents

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More information

Preface

This elementary book is intended for advanced undergraduates or anyone on a higher level who wants to learn the basic facts of p-adic analysis. We only assume the reader to have some standard knowledge of analysis and algebra.

In analysis (and outside it) the fields $\mathbb{R}$ and $\mathbb{C}$ play a central role. For several reasons people started to study the implications of replacing $\mathbb{R}$ or $\mathbb{C}$ by a more general object, viz. a field K with a complete valuation $| \ |$ comparable to the absolute value function (see Definition 1.1). Many such fields other than $\mathbb{R}$ or $\mathbb{C}$ exist, their valuations are all 'non-archimedean', i.e. they satisfy the 'strong triangle inequality' $|x + y| \leqslant \max(|x|, |y|)$. The analysis in and over non-archimedean valued fields K is known as ultrametric (non-archimedean, p-adic) analysis.

In this book we shall treat the basic facts of ultrametric analysis together to form an alternative 'one variable calculus course'. Thus, in K we shall consider familiar concepts such as continuity, differentiability, (power) series, integration, etc. However, the strong triangle inequality causes fascinating deviations from the 'classical analysis' (over $\mathbb{R}$ or $\mathbb{C}$); let us mention a few of them.

(i) A series Σa_n in K converges if $\lim_{n\to\infty} a_n = 0$. The power series $\Sigma x^n/n!$ of exp (if it makes sense at all) converges only on a disc strictly contained in the closed unit disc $\{x : |x| \leqslant 1\}$. Hence $\Sigma 1/n!$ diverges (but $\Sigma n!$ converges in many K).
(ii) K is not ordered. Yet it is possible to define 'square roots of positive elements' in a natural way. It may happen that $\sqrt{16} = 4$ but $\sqrt{25} = -5$.
(iii) K is not connected. In fact, each disc is open and closed; each point of a disc is a centre; $\{x : |x| = 1\}$ is not the boundary of $\{x : |x| < 1\}$.
(iv) Liouville's theorem (a bounded analytic function $K \to K$ is constant) holds if and only if K is *not* locally compact.
(v) For each pair of continuous functions $f_1, f_2 : K^2 \to K$ there exists an $F : K^2 \to K$ such that $\partial F/\partial x = f_1$, $\partial F/\partial y = f_2$.

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During the past few decades ultrametric analysis has grown from a relatively small and remote area maintained by a few enthusiastic pioneers to a widely recognized and mature discipline. However, since there are so many parts of mathematics for which a corresponding ultrametric theory exists or is at least conceivable, the term 'discipline' is somewhat misleading. In fact, in the 1980 classification scheme of the *Mathematical Reviews* explicit references to p-adic and non-archimedean theories are headed under Number theory; Algebraic number theory, field theory and polynomials; Algebraic geometry; Group theory and generalizations; Topological groups and Lie groups; Real functions; Functions of a complex variable; Several complex variables and analytic spaces and Functional analysis. It is clear that in an elementary book like this we cannot go into all these different branches, each one of which has its own origin, motivation, problems, language and applications. On the contrary, this book is written deliberately from no specific background at all but takes a 'naive' approach.

Depending on his or her point of view the beginner may use this book (at least the first three chapters of it) either as a 'main entrance' to the various specialistic theories or as an introduction to p-adic analysis in its own right. The specialist should not look for deep theorems. Yet I hope that he or she will appreciate having the elementary facts together with the proofs collected into a single volume.

For the reader's convenience many exercises of varying degree of difficulty have been included. They are meant for training purposes and also to indicate interesting by-paths. Exercises leading to results used in the main theory are marked*. In Appendix A we discuss some themes that have something to do with the subject but do not fit into the main text, mostly because of their functional analytic character. Appendix B contains several basic definitions and facts needed in the book and is meant as a refresher course, only to be consulted if the reader needs part of it.

A small list of books is given for further reading; in some of these one may find extensive lists of references. Perhaps closest to our book is Mahler (1980), whereas Bachman (1964) is also elementary but moves towards algebraic theory. In Amice (1975) one can find more about analytic functions. Monna (1970) and van Rooij (1978) treat functional analysis. Iwasawa (1972) considers special functions of interest for number theory. Koblitz (1977), after an elementary start, moves towards the p-adic zeta functions and Dwork's theory. More advanced studies and applications in algebraic number theory and algebraic geometry can be found in Koblitz (1980).

Only occasionally – in cases where a theorem is generally known as such – names of inventors are given. I have also added names to results that have not been published before and were pointed out to me by informal communication.

I am grateful to Lucien van Hamme and Arnoud van Rooij for their helpful comments and the stimulating effect they had on me.

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