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CHAPTER I

THE MISTAKE, THE HOWLER AND THE FALLACY

All mathematicians are wrong at times. In most cases the error is simply a MISTAKE, of little significance and, one hopes, of even less consequence. Its cause may be a momentary aberration, a slip in writing, or the misreading of earlier work. For present purposes it has no interest and will be ignored.

The HOWLER in mathematics is not easy to describe, but the term may be used to denote an error which leads *innocently* to a *correct* result. By contrast, the FALLACY leads by *guile* to a *wrong* but plausible conclusion.

One or two simple examples will illustrate the use of these terms. The first is a howler of a kind that many people, here and abroad, might think the British system of units only too well deserves:

To make out a bill:

$\frac{1}{4}$ lb. butter	@ 2*s*. 10*d*. per lb.
$2\frac{1}{2}$ lb. lard	@ 10*d*. per lb.
3 lb. sugar	@ $3\frac{1}{4}d$. per lb.
6 boxes matches	@ 7*d*. per dozen.
4 packets soap-flakes	@ $2\frac{1}{2}d$. per packet.

(Who but an examiner would include four packets of soap-flakes in such an order?)

The solution is

$$8\tfrac{1}{2}d. + 2s.\ 1d. + 9\tfrac{3}{4}d. + 3\tfrac{1}{2}d. + 10d. = 4s.\ 8\tfrac{3}{4}d.$$

One boy, however, avoided the detailed calculations and simply added all the prices on the right:

$$2s.\ 10d. + 10d. + 3\tfrac{1}{4}d. + 7d. + 2\tfrac{1}{2}d. = 4s.\ 8\tfrac{3}{4}d.$$

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The innocence of the pupil and the startling accuracy of the answer raise the calculation to the status of howler.

The emphasis of this book is on the fallacy, though a selection of howlers (leading, sometimes, to curious generalisations) will be given in the last chapter. Two synthetic howlers may, however, be added here before we proceed to the real business:

(i) *To prove the formula*

$$a^2 - b^2 = (a-b)(a+b).$$

Consider the quotient

$$\frac{a^2-b^2}{a-b}.$$

'Cancel' an a and a b:

$$\frac{a^{\not 2}-b^{\not 2}}{\not a-\not b},$$

and then 'cancel' the minus 'into' the minus:

$$\frac{+}{a^{\not 2}\not- b^{\not 2}}{\not a\not- \not b}.$$

The result is $a+b$.

(ii) *To simplify* $\frac{26}{65}$ *and* $\frac{16}{64}$.

The correct answers are obtained by cancelling:

$$\frac{2\not 6}{\not 6 5} = \frac{2}{5}$$

and

$$\frac{1\not 6}{\not 6 4} = \frac{1}{4}.$$

Consider next a typical fallacy:

To prove that every triangle is isosceles.

Let ABC (Fig. 1) be a given triangle. It is required to prove that AB is *necessarily* equal to AC.

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If the internal bisector of the angle A meets BC in D, then, by the angle-bisector theorem,

$$\frac{DB}{AB} = \frac{DC}{AC}.$$

Now
$$\begin{aligned}\angle ADB &= \angle ACD + \angle CAD \\ &= C + \tfrac{1}{2}A,\end{aligned}$$

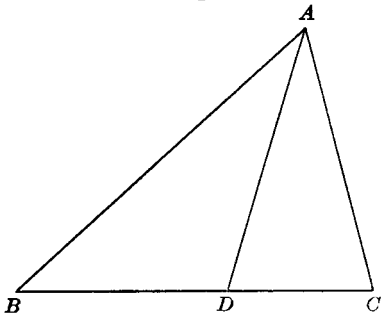

Fig. 1

so that, by the sine rule applied to the triangle ADB,

$$\begin{aligned}\frac{DB}{AB} &= \frac{\sin BAD}{\sin ADB} \\ &= \frac{\sin \tfrac{1}{2}A}{\sin (C + \tfrac{1}{2}A)}.\end{aligned}$$

Further,
$$\begin{aligned}\angle ADC &= \angle ABD + \angle BAD \\ &= B + \tfrac{1}{2}A,\end{aligned}$$

so that
$$\frac{DC}{AC} = \frac{\sin \tfrac{1}{2}A}{\sin (B + \tfrac{1}{2}A)}.$$

Hence
$$\frac{\sin \tfrac{1}{2}A}{\sin (C + \tfrac{1}{2}A)} = \frac{\sin \tfrac{1}{2}A}{\sin (B + \tfrac{1}{2}A)}.$$

Moreover $\sin \tfrac{1}{2}A$ is not zero, since the angle A is not zero,

and so
$$\sin (C + \tfrac{1}{2}A) = \sin (B + \tfrac{1}{2}A),$$

or
$$C + \tfrac{1}{2}A = B + \tfrac{1}{2}A,$$

or
$$C = B.$$

The triangle is therefore isosceles.

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The analysis of this fallacy may be used to illustrate standard features which we shall meet often.

To begin with, of course, the actual error must be detected. (The false step here follows the line

$$\sin(C+\tfrac{1}{2}A) = \sin(B+\tfrac{1}{2}A).$$

Equality of sine need not mean equality of angle.) But this, in a good fallacy, is only a part of the interest, and the lesser part at that.

The second stage is to effect, as it were, a *reconciliation statement* in which (i) the correct deduction is substituted for the false and, when possible, (ii) the discrepancy between the wrong and the correct theorems is accounted for in full.

Thus the step

$$\sin(C+\tfrac{1}{2}A) = \sin(B+\tfrac{1}{2}A)$$

leads not only to the stated conclusion

$$C+\tfrac{1}{2}A = B+\tfrac{1}{2}A,$$

but also to the alternative

$$C+\tfrac{1}{2}A = 180^\circ - (B+\tfrac{1}{2}A),$$

or

$$A+B+C = 180^\circ.$$

The necessity for the angles B, C to be equal is negatived by the fact that the sum of the angles A, B, C is *always* 180°. The error is therefore found and the correct version substituted.

In the work that follows we shall usually begin with a straightforward statement of the fallacious argument, following it with an exposure in which the error is traced to its most elementary source. It will be found that this process may lead to an analysis of unexpected depth, particularly in the first geometrical fallacy given in the next chapter.

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CHAPTER II

FOUR GEOMETRICAL FALLACIES ENUNCIATED

There are four well-known fallacies whose statement requires only the familiar theorems of elementary (schoolboy) geometry. They are propounded without comment in this chapter, the analysis, which becomes very fundamental, being postponed. The reader will doubtless wish to detect the errors for himself before proceeding to their examination in the next chapter.

¶ 1. THE FALLACY OF THE ISOSCELES TRIANGLE.

To prove that every triangle is isosceles.

GIVEN: A triangle ABC (Fig. 2).

REQUIRED: To prove that, necessarily,

$$AB = AC.$$

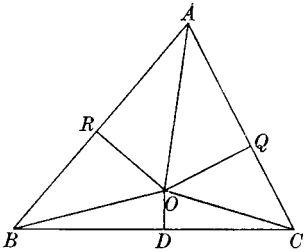

Fig. 2

CONSTRUCTION: Let the internal bisector of the angle A meet the perpendicular bisector of BC at O. Draw OD, OQ, OR perpendicular to BC, CA, AB respectively.

PROOF: Since

$$\begin{aligned} DO &= DO \\ DB &= DC \\ \angle ODB &= \angle ODC \\ \therefore\ \triangle ODB &\equiv \triangle ODC \qquad \text{(SAS)*} \\ \therefore\ OB &= OC. \end{aligned}$$

* Notation such as SAS is used as an abbreviation for 'having two sides and the included angle equal'. The symbol $\equiv$ denotes congruence.

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Also $AO = AO$,

$\angle RAO = \angle QAO$

$\angle ARO = \angle AQO$

$\therefore \triangle ARO \equiv \triangle AQO$ (ASA)

$\therefore AR = AQ$

and $OR = OQ$.

Hence, in triangles OBR, OCQ,

$\angle ORB = \angle OQC =$ right angle,

$OB = OC$ (proved)

$OR = OQ$ (proved)

$\therefore \triangle ORB \equiv \triangle OQC$ (rt. $\angle$, H, S)

$\therefore RB = QC$.

Finally, $AB = AR + RB$

$= AQ + QC$ (proved)

$= AC$. Q.E.D.

¶ 2. The Fallacy of the Right Angle.

To prove that every angle is a right angle.

GIVEN: A square $ABCD$ and a line BE drawn outwards from the square so that $\angle ABE$ has a given value, assumed obtuse (Fig. 3). (If the value is acute, take $\angle ABE$ to be its supplement.)

REQUIRED: To prove that

$\angle ABE =$ a right angle.

CONSTRUCTION: Let P, Q be the middle points of CD, AB. Take BE equal in length to a side of the square, and let the perpendicular bisector of DE meet PQ, produced if necessary, at O.

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PROOF: By symmetry, PQ is the perpendicular bisector of CD and of AB.

Consider the triangles ORD, ORE:

$$OR = OR$$
$$\angle ORD = \angle ORE \quad \text{(construction)}$$
$$RD = RE \quad \text{(construction)}$$
$$\therefore \ \triangle ORD \equiv \triangle ORE. \quad \text{(SAS)}$$

In particular, $OD = OE.$

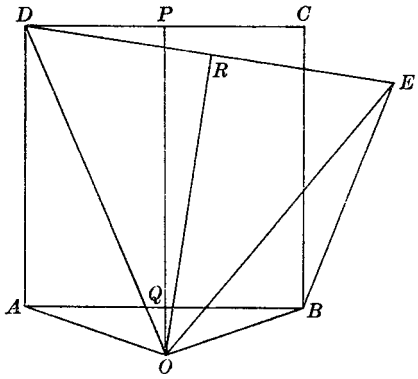

Fig. 3

Consider the triangles OQA, OQB:

$$OQ = OQ$$
$$\angle OQA = \angle OQB \quad \text{(perpendicular bisector)}$$
$$QA = QB \quad \text{(construction)}$$
$$\therefore \ \triangle OQA \equiv \triangle OQB. \quad \text{(SAS)}$$

In particular, $OA = OB$

and (for later reference)

$$\angle OAB = \angle OBA.$$

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Consider the triangles OAD, OBE:

$$OA = OB \qquad \text{(proved)}$$
$$AD = BE \quad (BE = \text{side of square})$$
$$OD = OE \qquad \text{(proved)}$$
$$\therefore\ \triangle OAD \equiv \triangle OBE. \qquad \text{(SSS)}$$

In particular,

$$\angle OAD = \angle OBE.$$

Now the point O on PQ may be either (i) between P, Q; (ii) at Q; (iii) beyond Q (as in Fig. 3).

In case (i)

$$\angle ABE = \angle OBE + \angle OBA$$
$$= \angle OAD + \angle OAB \qquad \text{(proved)}$$
$$= \text{right angle.}$$

In case (ii)

$$\angle ABE \equiv \angle OBE$$
$$= \angle OAD \qquad \text{(proved)}$$
$$\equiv \angle BAD$$
$$= \text{right angle.}$$

In case (iii)

$$\angle ABE = \angle OBE - \angle OBA$$
$$= \angle OAD - \angle OAB \qquad \text{(proved)}$$
$$= \text{right angle.}$$

Hence, in all three cases,

$$\angle ABE = \text{one right angle.}$$

¶ 3. The Trapezium Fallacy.

To prove that, if $ABCD$ is a quadrilateral in which $AB = CD$, then AD is necessarily parallel to BC.

GIVEN: A quadrilateral $ABCD$ (Fig. 4) in which $AB = CD$.

REQUIRED: To prove that AD is parallel to BC.

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CONSTRUCTION: Draw the perpendicular bisectors of AD, BC. If they are parallel, the theorem is proved. If not, let them meet in O, which may be outside (Fig. 4 (i)) or inside (Fig. 4 (ii)) the quadrilateral.

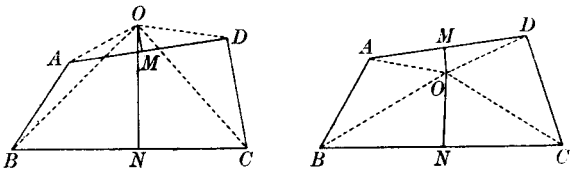

Fig. 4

PROOF: The perpendicular bisector of a line is the locus of points equally distant from its ends. Hence

$$OA = OD$$

$$OB = OC.$$

Compare the triangles OAB, ODC.

$$OA = OD \qquad \text{(proved)}$$

$$OB = OC \qquad \text{(proved)}$$

$$AB = DC \qquad \text{(given)}$$

$$\therefore\ \triangle OAB \equiv \triangle ODC \qquad \text{(SSS)}$$

$$\therefore\ \angle OAB = \angle ODC.$$

Also, by isosceles triangles,

$$\angle OAD = \angle ODA. \qquad (OA = OD)$$

Hence, by subtraction (Fig. 4.(i)) or addition (Fig. 4 (ii)),

$$\angle BAD = \angle CDA.$$

Similarly

$$\angle OBA = \angle OCD \quad (\triangle OAB \equiv \triangle ODC)$$

and $$\angle OBC = \angle OCB. \qquad (OB = OC)$$

Hence, by addition,

$$\angle ABC = \angle DCB.$$

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Thus

$$\angle BAD + \angle ABC = \angle CDA + \angle DCB.$$

But the sum of these four angles is four right angles, since they are the angles of the quadrilateral $ABCD$. Hence

$$\angle BAD + \angle ABC = 2 \text{ right angles.}$$

Also these are interior angles for the lines AD, BC with transversal AB.

Thus AD is parallel to BC.

¶ 4. The Fallacy of the Empty Circle.

To prove that every point inside a circle lies on its circumference.

GIVEN: A circle of centre O and radius r, and an arbitrary point P inside it (Fig. 5).

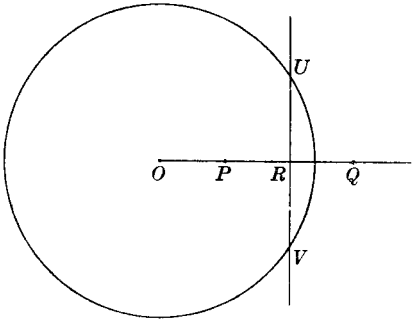

Fig. 5

REQUIRED: To prove that P lies on the circumference.

CONSTRUCTION: Let Q be the point on OP produced beyond P such that

$$OP.OQ = r^2,$$

and let the perpendicular bisector of PQ cut the circle at U, V. Denote by R the middle point of PQ.

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