CAMBRIDGE TRACTS IN MATHEMATICS

82. *Polycyclic groups*

DANIEL SEGAL

Reader in Mathematics
University of Manchester Institute of Science & Technology

Polycyclic groups

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This book is dedicated to my parents

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Preface

Nothing is simpler than a cyclic group. So if we build a group, starting from the identity, by a finite number of iterated extensions with cyclic groups, we would expect its structure to be pretty transparent. Such a group is called *polycyclic*.

In one sense, of course, polycyclic groups do have a transparent structure. But in the last few years, some remarkably intricate mathematics has been brought to bear on the study of these groups. Of course, the question of whether the end justifies the means is ultimately a matter of personal taste; to me, the picture which has begun to emerge is an attractive one. Working in this subject has given me a lot of pleasure, and if a little of that gets across to the reader of this book then the effort of writing it will have been well worth while.

This is not an encyclopaedic work on polycyclic groups. A number of thoroughly deserving topics have been omitted altogether, or merely touched on in the text (some of these, with references, are mentioned in the appendix). My guiding aim has simply been to present a connected account of some interesting mathematics, and throughout I have laid more stress on the ideas than on the results. In consequence, some of the results are given in lesser generality than they might be, and some of the proofs are leisurely where they could have been slick. Having been thus exposed to the basic techniques, the newcomer to this subject should be in a position to invent his/her own improvements.

More specifically, the purpose of the book is twofold. The earlier chapters are intended to provide a convenient and self-contained reference for the body of 'classical' results on polycyclic groups; Chapters 1–5 form an introductory course, suitable for the beginning research student (perhaps leaving out sections D and E of Chapter 4). The second half of the book is an introduction to more advanced topics, including the 'isomorphism problem' (Chapters 6–8) and the recent finiteness theorem of Grunewald–Pickel–Segal regarding groups with isomorphic finite quotients (Chapters 9 and 10). The final chapter, by way of light relief, offers various examples of polycyclic groups to illustrate some of the themes discussed before.

The results of the later chapters depend heavily on the theory of linear groups, on algebraic number theory, and on the theory of algebraic groups. I have endeavoured to present the necessary material in a reasonably general form, in order to emphasize that our results are but minor applications of some powerful and important mathematics. Some algebraic number theory is also used in Chapters 2 and 4; rather than include proofs of the relevant elementary results, I refer the reader to standard textbooks for material which every algebraist should know. However, the important theorem of Schmidt–Chevalley on the congruence subgroup property of unit groups in rings of algebraic integers is proved in full, modulo 'standard' results, in section E of Chapter 4.

Few of the results in this book are really new; some of the arguments may be. References to the literature are given in the 'Notes' sections at the end of each chapter; but the absence of a reference for a specific result does not imply any claim of originality (it is quite likely due to the author's ignorance). However, I believe that most of the theory of Chapter 7 and the last part of Chapter 8 are fairly original; the argument of Chapter 10 is also rather an improvement on the published version.

The 'Notes' sections also give fairly copious suggestions for further reading. These are particularly important in the case of Chapter 6, which deals with certain aspects of torsion-free finitely generated nilpotent groups (called '$\mathfrak{T}$-groups' throughout the text); there is an extensive and elegant theory of these groups which needs a book to itself – such books exist, and should be read as a necessary complement to this one by anyone wishing to learn about polycyclic groups.

Exercises are scattered liberally throughout the book. These are often an essential part of the text; with the generous hints they are almost all supposed to be very easy (the reader who wants to think harder should in the first instance ignore the hints!)

It may be helpful if I suggest here some 'subsequences' of the book which tell a reasonably connected story.

Core course Chapters 1, 2 and 5

Second course Chapter 3, Chapter 4 (sections A, B and C), Chapter 11.

Advanced course Chapter 4, Chapter 6, Chapter 7 (section A), Chapter 8 (sections A, B and C).

Special topics Chapter 7, Chapter 8 (section D), Chapters 9 and 10.

Acknowledgements

Anyone who works on polycyclic groups owes a debt to the 'founding fathers': Kurt Hirsch, who started the whole thing off in the 1930's, Reinhold Baer, A.I. Mal'cev and Philip Hall. Most of what I know about the subject I learnt from Bert Wehrfritz, Otto Kegel, Karl Gruenberg, Jim Roseblade and Fred Pickel; I thank them all for teaching me. I must also acknowledge a special debt to my friend and long-time collaborator Fritz Grunewald: what I have learnt from him is not easily summarised, but it has affected my whole outlook on mathematics.

Bert Wehrfritz read the whole manuscript with great care, and I am extremely grateful to him for his many helpful comments, resulting in a number of corrections and improvements. A word of appreciation is due to the audiences in Queen Mary College, Bielefeld and Manchester who sat through my lectures on parts of this work and helped me to sort it all out; particular thanks are due to Dave Warhurst, whose notes of a course I gave in Manchester formed the skeleton for the later parts of the book, and to Mrs Margaret Hillock for help with the typing.

Finally, I wish to thank the kindly editor, Dr Roseblade, for accepting this book for the Cambridge Tracts.

Notation

$A \leq B$	A is a subgroup of B
$A < B$	A is a proper subgroup of B
$A \lhd B$	A is a normal subgroup of B (groups); A is an ideal of B (rings)
$A \leq_f B$ ($A \lhd_f B$)	A is subgroup (normal subgroup) of finite index in B
$\langle X \rangle$	group generated by the set X
$G^n = \langle g^n \mid g \in G \rangle$	if G is a group and n is a positive integer
$A \times B$	direct product of A and B
$\mathrm{Dr}_{i \in I} A_i$	restricted direct product of the A_i
$\prod_{i \in I} A_i$	Cartesian product of the A_i
$\bigoplus_{i \in I} A_i$	direct sum of the A_i
$A] B$	semi-direct product of (normal subgroup) A by B
$x^y = y^{-1}xy$	if x and y belong to the same group
$[x, y] = x^{-1}x^y$	
$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$ for $n > 2$	
$[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle$	
$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n]$ for $n > 2$	
G'	derived group of G
$\gamma_i(G)$	ith term of the lower central series of G
$G^{(n)}$	nth term of the derived series of G
$\zeta_i(G)$	ith term of the upper central series of G
$\mathrm{Fitt}(G)$	Fitting subgroup of G
$\mathrm{Aut}\, G$	Automorphism group of G
$M_n(R)$	ring of $n \times n$ matrices over ring R
$GL_n(R)$	group of invertible matrices in $M_n(R)$
$D_n(R)$	group of diagonal matrices in $GL_n(R)$
$\mathrm{Tr}_n(R)$	group of upper-triangular matrices in $GL_n(R)$
$\mathrm{Tr}_1(n, R)$	group of matrices in $\mathrm{Tr}_n(R)$ with all diagonal entries 1
R^*	group of units of ring R
R^+	additive group of ring R
$\mathrm{End}_R(E)$	endomorphism ring of R-module E
$\mathrm{Aut}_R(E)$	automorphism group of R-module E
$C_G(X) = \{g \in G \mid x^g = x \ \forall x \in X\}$, the centralizer of X in G	

$N_G(X) = \{g \in G \mid X^g = X\}$, the normalizer of X in G

$\mathbb{N}$	the natural numbers (excluding zero)
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rational numbers
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{Z}_p$	the p-adic integers
$\mathbb{Q}_p$	the p-adic numbers
C_∞	the infinite cyclic group
C_n	the cyclic group of order n
S_n	the symmetric group of degree n (sometimes identified with the group of all $n \times n$ permutation matrices in $GL_n(\mathbb{Z})$)