

Cambridge University Press

0521020387 - Group Theoretical Methods and Applications to Molecules and Crystals

Shoon K. Kim

Frontmatter

[More information](#)

This book explains the basic aspects of symmetry groups as applied to problems in physics and chemistry using an approach pioneered and developed by the author. The symmetry groups and their representations are worked out explicitly, eliminating the unduly abstract nature of group theoretical methods.

The author has systematized the wealth of knowledge on symmetry groups that has accumulated during the century since Fedrov discovered the 230 space groups. All space groups, unitary as well as anti-unitary, are reconstructed from the algebraic defining relations of the point groups. The matrix representations are determined through the projective representations of the point groups. The representations of the point groups are subduced by the representations of the rotation group. The correspondence theorem on basis functions belonging to a representation is introduced to form the general expression for the symmetry-adapted linear combinations of equivalent basis functions with respect to a point group. This is then applied to form molecular orbitals and symmetry coordinates of vibration of a molecule or a crystal and the energy band eigenfunctions of the electrons in a crystal. The book assumes only an elementary knowledge of quantum mechanics. Numerous applications of the theorems are described to aid understanding.

This work will be of great interest to graduate students and professionals in solid state physics, chemistry, mathematics and geology and to those who are interested in magnetic crystal structures.

Cambridge University Press

0521020387 - Group Theoretical Methods and Applications to Molecules and Crystals

Shoon K. Kim

Frontmatter

[More information](#)

To 'Eomma'

Cambridge University Press

0521020387 - Group Theoretical Methods and Applications to Molecules and Crystals

Shoon K. Kim

Frontmatter

[More information](#)

Group Theoretical Methods and Applications to Molecules and Crystals

SHOON K. KIM

Temple University



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
0521020387 - Group Theoretical Methods and Applications to Molecules and Crystals
Shoon K. Kim
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521640626

© Shoon K Kim 1999

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 1999
This digitally printed first paperback version 2005

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Kim, Shoon Kyung, 1920–
Group theoretical methods and applications to molecules and crystals/by Shoon K. Kim.
p. cm.

Includes bibliographical references and index.

ISBN 0 521 64062 8

1. Group theory. 2. Chemistry, Physical and theoretical – Mathematics. 3. Crystallography, Mathematical.

I. Title. QD455.3.G75K56 1999

512'.2–dc21 98–26493 CIP

ISBN-13 978-0-521-64062-6 hardback

ISBN-10 0-521-64062-8 hardback

ISBN-13 978-0-521-02038-1 paperback

ISBN-10 0-521-02038-7 paperback

Contents

	<i>Preface</i>	page xv
	<i>List of symbols</i>	xvii
1	Linear transformations	
1.1	Vectors	1
1.2	Linear transformations and matrices	2
	1.2.1 Functions of a matrix	5
	1.2.2 Special matrices	6
	1.2.3 Direct products of matrices	6
	1.2.4 Direct sums of matrices	7
1.3	Similarity transformations	8
	1.3.1 Functions of a matrix (revisited)	8
1.4	The characteristic equation of a matrix	9
	1.4.1 Diagonalizability and projection operators	10
1.5	Unitary transformations and normal matrices	12
	1.5.1 Examples of normal matrices	13
1.6	Exercises	14
2	The theory of matrix transformations	
2.1	Involutorial transformations	17
2.2	Application to the Dirac theory of the electron	20
	2.2.1 The Dirac γ -matrices	20
	2.2.2 The Dirac plane waves	21
	2.2.3 The symmetric Dirac plane waves	23
2.3	Intertwining matrices	24
	2.3.1 Idempotent matrices	26
2.4	Matrix diagonalizations	27
2.5	Basic properties of the characteristic transformation matrices	30
2.6	Construction of a transformation matrix	31
2.7	Illustrative examples	34
3	Elements of abstract group theory	
3.1	Group axioms	37
	3.1.1 The criterion for a finite group	38
	3.1.2 Examples of groups	38
3.2	Group generators for a finite group	40
	3.2.1 Examples	42
3.3	Subgroups and coset decompositions	43
	3.3.1 The criterion for subgroups	44

vi	<i>Contents</i>	
	3.3.2 Lagrange's theorem	44
3.4	Conjugation and classes	45
	3.4.1 Normalizers	45
	3.4.2 The centralizer	46
	3.4.3 The center	47
	3.4.4 Classes	47
3.5	Isomorphism and homomorphism	48
	3.5.1 Examples	49
	3.5.2 Factor groups	50
3.6	Direct products and semidirect products	51
4	Unitary and orthogonal groups	
4.1	The unitary group $U(n)$	53
	4.1.1 Basic properties	53
	4.1.2 The exponential form	54
4.2	The orthogonal group $O(n, c)$	55
	4.2.1 Basic properties	55
	4.2.2 Improper rotation	56
	4.2.3 The real orthogonal group $O(n, r)$	57
	4.2.4 Real exponential form	58
4.3	The rotation group in three dimensions $O(3, r)$	58
	4.3.1 Basic properties of rotation	58
	4.3.2 The conjugate rotations	62
	4.3.3 The Euler angles	63
5	The point groups of finite order	
5.1	Introduction	66
	5.1.1 The uniaxial group C_n	67
	5.1.2 Multiaxial groups. The equivalence set of axes and axis-vectors	67
	5.1.3 Notations and the multiplication law for point operations	68
5.2	The dihedral group D_n	73
5.3	Proper polyhedral groups P_o	75
	5.3.1 Proper cubic groups, T and O	77
	5.3.2 Presentations of polyhedral groups	79
	5.3.3 Subgroups of proper point groups	82
	5.3.4 Theorems on the axis-vectors of proper point groups	83
5.4	The Wyle theorem on proper point groups	85
5.5	Improper point groups	86
	5.5.1 General discussion	86
	5.5.2 Presentations of improper point groups	88
	5.5.3 Subgroups of point groups of finite order	90
5.6	The angular distribution of the axis-vectors of rotation for regular polyhedral groups	91
	5.6.1 General discussion	91
	5.6.2 Icosahedral group Y	93
	5.6.3 Buckminsterfullerene C_{60} (buckyball)	97
5.7	Coset enumeration	98

Contents

vii

6	Theory of group representations	
6.1	Hilbert spaces and linear operators	102
	6.1.1 Hilbert spaces	102
	6.1.2 Linear operators	103
	6.1.3 The matrix representative of an operator	105
6.2	Matrix representations of a group	107
	6.2.1 Homomorphism conditions	107
	6.2.2 The regular representation	109
	6.2.3 Irreducible representations	110
6.3	The basis of a group representation	112
	6.3.1 The carrier space of a representation	112
	6.3.2 The natural basis of a matrix group	114
6.4	Transformation of functions and operators	115
	6.4.1 General discussion	115
	6.4.2 The group of transformation operators	117
	6.4.3 Transformation of operators under $G = \{\hat{R}\}$	119
6.5	Schur's lemma and the orthogonality theorems on irreducible representations	120
6.6	The theory of characters	125
	6.6.1 Orthogonality relations	125
	6.6.2 Frequencies and irreducibility criteria	126
	6.6.3 Group functions	127
6.7	Irreducible representations of point groups	128
	6.7.1 The group C_n	129
	6.7.2 The group D_n	129
	6.7.3 The group T	131
	6.7.4 The group O	133
	6.7.5 The improper point groups	135
6.8	Properties of irreducible bases	135
	6.8.1 The orthogonality of basis functions	135
	6.8.2 Application to perturbation theory	136
6.9	Symmetry-adapted functions	138
	6.9.1 Generating operators	138
	6.9.2 The projection operators	141
6.10	Selection rules	144
7	Construction of symmetry-adapted linear combinations based on the correspondence theorem	
7.1	Introduction	149
7.2	The basic development	150
	7.2.1 Equivalent point space $S^{(n)}$	150
	7.2.2 The correspondence theorem on basis functions	152
	7.2.3 Mathematical properties of bases on $S^{(n)}$	153
	7.2.4 Illustrative examples of the SALCs of equivalent scalars	155
7.3	SALCs of equivalent orbitals in general	158
	7.3.1 The general expression of SALCs	158
	7.3.2 Two-point bases and operator bases	160
	7.3.3 Notations for equivalent orbitals	161

viii	<i>Contents</i>	
	7.3.4 Alternative elementary bases	161
	7.3.5 Illustrative examples	162
7.4	The general classification of SALCs	166
	7.4.1 $D^{(A)}$ SALCs from the equivalent orbitals $\in D^{(A)} \times \Delta^{(n)}$	167
7.5	Hybrid atomic orbitals	170
	7.5.1 The σ -bonding hybrid AOs	171
	7.5.2 General hybrid AOs	173
7.6	Symmetry coordinates of molecular vibration based on the correspondence theorem	174
	7.6.1 External symmetry coordinates of vibration	175
	7.6.2 Internal vibrational coordinates	177
	7.6.3 Illustrative examples	179
8	Subduced and induced representations	
8.1	Subduced representations	188
8.2	Induced representations	189
	8.2.1 Transitivity of induction	191
	8.2.2 Characters of induced representations	191
	8.2.3 The irreducibility condition for induced representations	191
8.3	Induced representations from the irreps of a normal subgroup	193
	8.3.1 Conjugate representations	193
	8.3.2 Little groups and orbits	194
	8.3.3 Examples	195
8.4	Irreps of a solvable group by induction	197
	8.4.1 Solvable groups	197
	8.4.2 Induced representations for a solvable group	198
	8.4.3 Case I (reducible)	199
	8.4.4 Case II (irreducible)	201
	8.4.5 Examples	202
8.5	General theorems on induced and subduced representations and construction of unirreps via small representations	203
	8.5.1 Induction and subduction	203
	8.5.2 Small representations of a little group	205
	8.5.3 Induced representations from small representations	206
9	Elements of continuous groups	
9.1	Introduction	209
	9.1.1 Mixed continuous groups	210
9.2	The Hurwitz integral	211
	9.2.1 Orthogonality relations	215
9.3	Group generators and Lie algebra	215
9.4	The connectedness of a continuous group and the multivalued representations	219
10	The representations of the rotation group	
10.1	The structure of $SU(2)$	224
	10.1.1 The generators of $SU(2)$	224
	10.1.2 The parameter space Ω' of $SU(2)$	226

<i>Contents</i>		ix
10.1.3	Spinors	228
10.1.4	Quaternions	228
10.2	The homomorphism between $SU(2)$ and $SO(3, r)$	229
10.3	Unirreps $D^{(j)}(\boldsymbol{\theta})$ of the rotation group	232
10.3.1	The homogeneity of $D^{(j)}(S)$	233
10.3.2	The unitarity of $D^{(j)}(S)$	234
10.3.3	The irreducibility of $D^{(j)}(\boldsymbol{\theta})$	234
10.3.4	The completeness of the unirreps $\{D^{(j)}(\boldsymbol{\theta}); j = 0, \frac{1}{2}, 1, \dots\}$	235
10.3.5	Orthogonality relations of $D^{(j)}(\boldsymbol{\theta})$	235
10.3.6	The Hurwitz density function for $SU(2)$	236
10.4	The generalized spinors and the angular momentum eigenfunctions	237
10.4.1	The generalized spinors	237
10.4.2	The transformation of the total angular momentum eigenfunctions under the general rotation U_J	238
10.4.3	The vector addition model	240
10.4.4	The Clebsch–Gordan coefficients	241
10.4.5	The angular momentum eigenfunctions for one electron	245
11	Single- and double-valued representations of point groups	
11.1	The double-valued representations of point groups expressed by the projective representations	247
11.1.1	The projective set of a point group	247
11.1.2	The orthogonality relations for projective unirreps	248
11.2	The structures of double point groups	250
11.2.1	Defining relations of double point groups	250
11.2.2	The structure of the double dihedral group D'_n	251
11.2.3	The structure of the double octahedral group O'	253
11.3	The unirreps of double point groups expressed by the projective unirreps of point groups	257
11.3.1	The uniaxial group C_∞	257
11.3.2	The group C_n	258
11.3.3	The group D_∞	258
11.3.4	The group D_n	260
11.3.5	The group O	261
11.3.6	The tetrahedral group T	263
12	Projective representations	
12.1	Basic concepts	266
12.2	Projective equivalence	268
12.2.1	Standard factor systems	269
12.2.2	Normalized factor systems	270
12.2.3	Groups of factor systems and multipliers	271
12.2.4	Examples of projective representations	272
12.3	The orthogonality theorem on projective irreps	274
12.4	Covering groups and representation groups	276
12.4.1	Covering groups	276
12.4.2	Representation groups	278

x	<i>Contents</i>	
12.5	Representation groups of double point groups	279
	12.5.1 Representation groups of double proper point groups P'	279
	12.5.2 Representation groups of double rotation–inversion groups P'_i	281
12.6	Projective unirreps of double rotation–inversion point groups P'_i	283
	12.6.1 The projective unirreps of $C'_{2r,i}$	285
	12.6.2 The projective unirreps of $D'_{n,i}$	286
	12.6.3 The projective unirreps of O'_i	287
13	The 230 space groups	
13.1	The Euclidean group in three dimensions $E^{(3)}$	289
13.2	Introduction to space groups	293
13.3	The general structure of Bravais lattices	295
	13.3.1 Primitive bases	295
	13.3.2 The projection operators for a Bravais lattice	298
	13.3.3 Algebraic expressions for the Bravais lattices	299
13.4	The 14 Bravais lattice types	302
	13.4.1 The hexagonal system H (D_{6i})	302
	13.4.2 The tetragonal system Q (D_{4i})	303
	13.4.3 The rhombohedral system RH (D_{3i})	306
	13.4.4 The orthorhombic system O (D_{2i})	307
	13.4.5 The cubic system C (O_i)	308
	13.4.6 The monoclinic system M (C_{2i})	309
	13.4.7 The triclinic system T (C_i)	311
	13.4.8 Remarks	311
13.5	The 32 crystal classes and the lattice types	313
13.6	The 32 minimal general generator sets for the 230 space groups	315
	13.6.1 Introduction	315
	13.6.2 The space groups of the class D_4	316
13.7	Equivalence criteria for space groups	318
13.8	Notations and defining relations	321
	13.8.1 Notations	321
	13.8.2 Defining relations of the crystal classes	322
13.9	The space groups of the cubic system	323
	13.9.1 The class T	326
	13.9.2 The class T_i ($=T_h$)	327
	13.9.3 The class O	329
	13.9.4 The class T_p ($=T_d$)	330
	13.9.5 The class O_i ($=O_h$)	331
13.10	The space groups of the rhombohedral system	332
	13.10.1 The class C_3	333
	13.10.2 The class C_{3i}	333
	13.10.3 The class D_3	333
	13.10.4 The class C_{3v}	334
	13.10.5 The class D_{3i} ($=D_{3d}$)	335
13.11	The hierarchy of space groups in a crystal system	336
	13.11.1 The cubic system	337
	13.11.2 The hexagonal system	337

<i>Contents</i>		xi
13.11.3	The rhombohedral system	337
13.11.4	The tetragonal system	338
13.12	Concluding remarks	338
14	Representations of the space groups	
14.1	The unirreps of translation groups	340
14.2	The reciprocal lattices	342
14.2.1	General discussion	342
14.2.2	Reciprocal lattices of the cubic system	344
14.2.3	The Miller indices	345
14.2.4	The density of lattice points on a plane	346
14.3	Brillouin zones	347
14.3.1	General construction of Brillouin zones	347
14.3.2	The wave vector point groups	348
14.3.3	The Brillouin zones of the cubic system	349
14.4	The small representations of wave vector space groups	352
14.4.1	The wave vector space groups \hat{G}_k	352
14.4.2	Small representations of \hat{G}_k via the projective representations of G_k	354
14.4.3	Examples of the small representations of \hat{G}_k	356
14.5	The unirreps of the space groups	358
14.5.1	The irreducible star	359
14.5.2	A summary of the induced representation of the space groups	360
15	Applications of unirreps of space groups to energy bands and vibrational modes of crystals	
15.1	Energy bands and the eigenfunctions of an electron in a crystal	362
15.2	Energy bands and the eigenfunctions for the free-electron model in a crystal	365
15.2.1	The notations for a small representation of \hat{G}_k	368
15.2.2	Example 1. A simple cubic lattice	368
15.2.3	Example 2. The diamond crystal	371
15.3	Symmetry coordinates of vibration of a crystal	377
15.3.1	General discussion	377
15.3.2	The small representations of the wave vector groups \hat{G}_k based on the equivalent Bloch functions	379
15.4	The symmetry coordinates of vibration for the diamond crystal	382
15.4.1	General discussion	382
15.4.2	Construction of the symmetry coordinates of vibration	385
16	Time reversal, anti-unitary point groups and their co-representations	
16.1	Time-reversal symmetry, classical	391
16.1.1	General introduction	391
16.1.2	The time correlation function	393
16.1.3	Onsager's reciprocity relation for transport coefficients	394

xii	<i>Contents</i>	
16.2	Time-reversal symmetry, quantum mechanical	396
16.2.1	General introduction	396
16.2.2	The properties of the time-reversal operator θ	400
16.2.3	The time-reversal symmetry of matrix elements of a physical quantity	401
16.3	Anti-unitary point groups	403
16.3.1	General discussion	403
16.3.2	The classification of ferromagnetics and ferroelectrics	407
16.4	The co-representations of anti-unitary point groups	409
16.4.1	General discussion	409
16.4.2	Three types of co-unirreps	410
16.5	Construction of the co-unirreps of anti-unitary point groups	413
16.5.1	G_s^e	414
16.5.2	C_n^e , C_n^q and C_n^u	415
16.5.3	D_n^e and D_n^q	417
16.5.4	The cubic groups	418
16.6	Complex conjugate representations	419
16.7	The orthogonality theorem on the co-unirreps	422
16.8	Orthogonality relations for the characters, the irreducibility condition and the type criteria for co-unirreps	424
16.8.1	Orthogonality relations for the characters of co-unirreps	424
16.8.2	Irreducibility criteria for co-unirreps	424
16.8.3	The type criterion for a co-unirrep	425
17	Anti-unitary space groups and their co-representations	
17.1	Introduction	428
17.2	Anti-unitary space groups of the first kind	430
17.2.1	The cubic system	432
17.2.2	The hexagonal system	433
17.2.3	The rhombohedral system	433
17.2.4	The tetragonal system	433
17.2.5	The orthorhombic system	433
17.2.6	The monoclinic system	435
17.2.7	The triclinic system	435
17.3	Anti-unitary space groups of the second kind	438
17.3.1	Illustrative examples	441
17.3.2	Concluding remarks	441
17.4	The type criteria for the co-unirreps of anti-unitary space groups and anti-unitary wave vector groups	442
17.5	The representation groups of anti-unitary point groups	445
17.6	The projective co-unirreps of anti-unitary point groups	450
17.6.1	Examples for the construction of the projective co-unirreps of H^2	458
17.7	The co-unirreps of anti-unitary wave vector space groups	460
17.7.1	Concluding remarks	463
17.8	Selection rules under an anti-unitary group	464
17.8.1	General discussion	464

Cambridge University Press

0521020387 - Group Theoretical Methods and Applications to Molecules and Crystals

Shoon K. Kim

Frontmatter

[More information](#)

<i>Contents</i>	xiii
17.8.2 Transitions between states belonging to different co-unirreps	465
17.8.3 Transitions between states belonging to the same co-unirrep	467
17.8.4 Selection rules under a gray point group	470
Appendix. Character tables for the crystal point groups	472
<i>References</i>	483
<i>Index</i>	487

Preface

This book is written for graduate students and professionals in physics, chemistry and in particular for those who are interested in crystal and magnetic crystal symmetries. It is mostly based on the papers written by the author over the last 20 years and the lectures given at Temple University. The aim of the book is to systematize the wealth of knowledge on point groups and their extensions which has accumulated over a century since Schönflies and Fedrov discovered the 230 space groups in 1895. Simple, unambiguous methods of construction for the relevant groups and their representations introduced in the book may overcome the abstract nature of the group theoretical methods applied to physical chemical problems.

For example, a unified approach to the point groups and the space groups is proposed. Firstly, a point group of finite order is defined by a set of the algebraic defining relations (or presentation) through the generators in Chapter 5. Then, by incorporating the translational degree of freedom into the presentations of the 32 crystallographic point groups, I have determined the 32 minimum general generator sets (MGGs) which generate the 230 space groups in Chapter 13. Their representations follow from a set of five general expressions of the projective representations of the point groups given in Chapter 12. It is simply amazing to see that the simple algebraic defining relations of point groups are so very far-reaching.

In almost all other textbooks or monographs on solid-state physics, the space groups may be tabulated, but without their derivations, as if they were ‘god-given’. The main reason could have been the lack of a simple method for the derivations. As a result, the group theoretical methods have been unnecessarily abstract in an age when students are very familiar with non-commuting physical quantities in quantum mechanics.

The book is self-sufficient even though some elementary knowledge of quantum mechanics is assumed. No previous knowledge of group theory is required. In providing the basic essentials, introductory examples are given prior to the theorems. Effort has been made to provide the simplest and easiest but rigorous proofs for any theorem described in the book. Applications are fully developed. Each chapter contains something new or different in approach that cannot be found in any other monograph. For example, even in the basic theory on matrix transformation given in Chapter 2, I have introduced an involutorial transformation into the Dirac theory of the electron and arrived at the Dirac plane wave solution in one step. This transformation is used frequently in later chapters. The transformation is further extended to a new general theory of matrix diagonalization that provides the transformation matrix as a polynomial of the matrix to be diagonalized. This theory is included for its usefulness even though it is somewhat mathematical.

Some further typical features of the book are worth mentioning here. In Chapter 5, I have introduced a faithful representation for a point group using the unit basis vectors of the coordinate system. This allows one to construct the multiplication table of any point

group, e.g. the octahedral group, with ease. A new unified system of classifications for the improper point groups and anti-unitary (or magnetic) point groups is introduced, using the fact that both the inversion and the time-reversal operator commute with all the point operations. This system is quite effective for describing their isomorphisms, and thereby greatly simplifies the construction of their matrix representations and co-representations in its entirety. In Chapter 7, I have introduced a simple correspondence theorem on the basis functions of a point group G and thereby developed a general method of constructing the symmetry-adapted linear combinations (SALCs) of equivalent basis functions with respect to G . It is then applied to construct SALCs of equivalent atomic orbitals and the symmetry coordinates of vibration for molecules and later for crystals in Chapter 15. This theory requires only knowledge of the elementary basis functions of the irreducible representation and does not require the matrix representation. This is in quite a contrast to the conventional projection operator method. The correspondence theorem is further extended to form the energy band eigenfunctions of the electron in a solid in Chapter 15. By incorporating the time-reversal symmetry into point groups, anti-unitary (magnetic) point groups are formed in Chapter 16. Analogously, 38 assemblies of MGGs for 1421 anti-unitary space groups are formed from the 32 MGGs of space groups in Chapter 17. Their co-representations are introduced and applied to the selection rules under the anti-unitary groups.

Once a reader is familiar with the basic aspects of the group theoretical methods given in Chapters 3, 4 and 5, the reader can pick and choose to read any applications in later chapters using the rest of the book as the built-in references. This is possible because each chapter is as self-contained as possible and also an effective numbering system is introduced for referring to the theorems, equations and figures given in the book. Numerous examples of the applications of theorems are given as illustrations. In some chapters, I introduced a simplified special proof for a theorem to help understanding, even though its general proof had been given in an earlier chapter. In particular, those who are interested in the applications to inorganic chemistry may directly start from Chapter 7 with minimum knowledge of the group theoretical methods. One of my colleagues, Professor S. Jansen-Varnum, used the theory of symmetry-adapted linear combinations based on the correspondence theorem described in Chapter 7 of my manuscript for teaching both undergraduate and graduate courses in inorganic chemistry.

Acknowledgment

I am very much indebted to many friends and colleagues for their help while I was writing this book. Firstly, I am very grateful to the chairpersons of the chemistry and physics departments, Dr G. Krow, Dr S. Wunder, Dr R. Salomon and E. Gawlinski. Special thanks are due to Dr D. J. Lee, the late Dr C. W. Pyun, Dr S. I. Choi, Dr S. Jansen-Varnum and Dr L. Mascavage for reviewing parts of the manuscript and to Dr K. S. Yun for valuable advice. I am also very grateful to Dr D. Titus for his help in scanning the figures into the manuscript. Whenever I had difficulty with a delicate sentence, Dr D. Dalton assisted me, so I express here my sincere gratitude for his help. Ms G. Basmajian typed the entire manuscript single-handedly. I am truly indebted to her patience and her typing skills in dealing with complicated mathematical equations.

Philadelphia, Pennsylvania

S. K. KIM

List of symbols

\in	belongs to, e.g. $g \in S$ means an element g belongs to a set S .
\forall	for all, e.g. $\forall g \in S$ means for all $g \in S$.
$*$	complex conjugate.
\sim	transpose, e.g. A^\sim is the transpose of the matrix A .
\dagger	adjoint or Hermitian adjoint, i.e. $A^\dagger = A^{*\sim}$.
\rightarrow	is mapped onto.
\leftrightarrow	one-to-one correspondence.
\otimes	direct product.
\oplus	direct sum.
\cap	set-theoretic intersection, e.g., $S_1 \cap S_2$ is the set common to the two sets S_1 and S_2 .
$\{ \}$	set of all elements.
$H < G$	H is a subgroup of a group G .
$H \triangleleft G$	H is an invariant subgroup of a group G .
$G_1 \times G_2$	the direct product group of two groups G_1 and G_2 .
$F \wedge H$	the semidirect product of two groups F and H , where F is invariant under H .
$F \simeq H$	Two groups F and H are isomorphic.