EXTENSIONS OF FIRST ORDER LOGIC

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Applying many-sorted logic to dynamic logic.

Preface.

This book considers various extensions of first order logic, giving detailed and elaborate treatment to many useful logical systems: second order logic (**SOL**), type theory (**RTT**, **ETT** and **FTT**), modal logic (**PML** and **FOML**), dynamic logic (**PDL**) and many-sorted logic (**MSL**). A substantial dose of logical perspective is also provided.

The second objective of this book is to pursue the thesis that most reasonable logical systems can be naturally translated into many-sorted first order logic. The thesis is maintained throughout the book, but only appears openly and explicitly in the last chapter. There, all the logic systems treated in the book are put in direct correspondence with many-sorted logic because this logic offers a unifying framework in which to place other logics. In itself, many-sorted logic is a natural logic for formalizing statements in a great variety of disciplines and it also has an efficient proof theory with a complete deductive calculus.

Currently, the proliferation of logics used in philosophy, computer science, artificial intelligence, mathematics and linguistics makes a working reduction of this variety an urgent issue. The aim is two-fold:

To be able to use only one deductive calculus and a unique theorem prover for all logics -*i.e.*, an **MSL** theorem prover;

To avoid the proofs of the metaproperties of the different existing logics by borrowing them from many-sorted logic.

The appeal of this approach is that it is so intuitive and easy that only common sense is needed to understand the construction. Besides, as the basic ingredients change, the recipe can be adapted and used to prepare different dishes. So with very little effort the results obtained are considerable. It is difficult to trace the development of this approach because almost every non-classical logic has found its standard counterpart at birth. Nevertheless, I like to credit most of the ideas involved in our current presentation to Henkin's paper "Completeness in the theory of types" (1950). However I do not want to be misleading; you are not going to find in this paper of 1950 anything like the translation of formulas into another formal language, or the open appearance of many-sorted logic. In connection with **SOL**, many-sorted logic appeared later, in Henkin's "Banishing the rule of substitution for functional variables" (1953), where a new second order calculus with the comprehension rule was presented. As noted in that paper, from this calculus it is possible to isolate the many-sorted calculus by leaving out the comprehension rule. Another remarkable discovery included in the 1953 paper is that we can weaken comprehension so that it applies only to a restricted class of formulas of our choice. This is of great help when treating the modal and dynamic logics where we restrict comprehension to the sets and relations defined by translations of formulas of **PML** or of **PDL**.

In Henkin 1950 paper the completeness of type theory is proved and the general models are presented. How did Henkin prove the completeness theorem for type theory? A very rapid answer to this question is: by changing the semantics and hence the logic. Roughly presented, the idea is very simple: The set of validities is so wide because our class of standard structures is too small. We have been very restrictive when requiring the relational universes of any model to contain all possible relations (where "possible" means in the background set theory used as metalanguage) and we have paid a high price for it. If we also allow nonstandard structures, and if we now interpret validity as being true in all general models, redefining all the semantic notions referring to this larger class of general structures, completeness (in both weak and strong senses), Löwenheim-Skolem, and all these theorems can be proven as in first order logic.

In addition to its usefulness, the general model's construction is far from being an ad hoc solution lacking naturalness and common sense. Throughout the pages of this book you will find good reasons for wondering whether the philosophy of standard structures is the only possible choice. The reasons are directly related to the following questions:

(1) Are we satisfied with the limitation on the class of models they require? Would it not be highly instructive to discover new and sensible models for a known existing theory?
(2) Don't we feel uneasy about crossing the border with set theory? Don't second order validities refer to the given set-theoretical environment instead of the logic in itself?
(3) Do we need all the expressive power they provide?
(4) Are we willing to pay the price that standard semantics demands?

Further motivation for using general models may be found in van Benthem's recent essay "The sources of complexity", where the author considers that with general semantics

> ...we achieve a moral rarity, being a combination of philosophical virtue with computational advantage,...

In fact, when considering the arguments used in the second chapter of this book, one can argue that the standard semantics is not logically adequate in the sense that it does not allow all logically possible interpretations of second order formulas as models because of the argument posed by Németi in the following form:

> We have to be placed in a set-theoretical universe, even assuming that there could be more than one such. Nevertheless, in the set theoretical universe you choose to be in, the GCH is either true or false. Assume it is true. Then, in every standard model for **SOL** the **SOL** formula φ expressing this hypothesis is true and so φ is valid. But since GCH is not derivable from ZFC, the result suggest that an interpretation $\mathcal{I}$ such that $\mathcal{I} \nvDash \varphi$ can not be excluded as "logically impossible". So, at least one $\mathcal{I}$ with $\mathcal{I} \vDash \neg\varphi$ is a logical possibility (by Paul Cohen's classical result). But such a model is not allowed in the standard semantics. So, we feel that the standard semantics does not include all logically possible worlds as models (we have to think about formulas, like GCH, which are both expressible in second order logic and independent from Zermelo-Fraenkel set theory). This argument is reinforced by the fact that there is an inexhaustible supply of independent formulas like GCH. In Henkin's general semantics many possibilities are restored as possible models; for instance, models with or without the GCH.

As you will see, the general model strategy is also used in modal logic and dynamic logic. Both logics are faithfully represented by many-sorted theories with a comprehension schema restricted to a definable subclass of many-sorted formulas.

A brief description of each chapter follows.

Chapter I. Standard Second Order Logic.
The first chapter is an introduction to standard second order logic with emphasis placed on the expressive power this logic provides. It consists of extending first order logic to second order logic by allowing quantification over sets and relations. Also mentioned are the model-theoretic counterparts of expressiveness, without overlooking the incompleteness result.

The definition of standard structure is given and the common relationships between standard structures are defined. Specifically, notions such as substructure, homomorphism, isomorphism and embedding are dealt with. A section is devoted to standard semantics where meaning is given to formulas of the formal language by introducing the related notions of satisfiability, validity and consequence. The question of what sets and relations are definable is then considered and a number of notions of definability are proposed. The chapter closes by proving a series of semantic theorems for standard second order logic, including the coincidence lemma, the substitution lemma and the isomorphism theorem.

Chapter II. Deductive Calculi.
A number of deductive calculus for second order logic are defined and soundness and incompleteness results are presented.

The chapter begins with an informal introduction defining what a calculus is, explaining the usual motivations for wanting a calculus, and stating the desired metaproperties of a calculus; ***i.e.***, that it would never drive us to erroneous reasoning and that it would also help us to derive all the consequences of a given set of hypotheses. Of the three calculi introduced, the first is a very simple extension of a first order calculus of sequents where the quantifiers' rules also cover the set and relation variables. It is defined for a second order language where equality among individuals is treated as in first order logic; that is, it is a primitive or logical symbol rather than a defined one. The calculus also contains the rules dealing with equality. Since we also have equality for predicates as primitive, we adjoin some equality rules for them and we will have the extensionality axiom as a rule without hypothesis. This calculus will not count as a second order calculus for many people. In fact, it is plain many-sorted, and in an imprecise way we can name it **MSL**. Adding comprehension or lambda rules we obtain proper second order calculus. Many deductions are done in full as exercises.

Another section is devoted to proving the soundness theorems for the three calculi. There is a very easy proof of incompleteness of **MSL** with standard semantics showing that $\exists xX\, Xx$ is valid in the standard semantics but is not derivable in **MSL**.

The chapter closes by proving the incompleteness result for second order logic with standard semantics, in both weak and strong senses, for any calculus.

Chapter III. Categoricity of Second Order Peano Arithmetic.
This chapter introduces Peano arithmetic in **SOL** and proves that with the standard semantics this theory is categorical; that is, any two second order models of Peano arithmetic are isomorphic. The question of non-standard models of first order Peano arithmetic is raised and put in direct correspondence with non-standard models of second order logic; quite different

meanings for the word "non-standard" which are, nevertheless, related.

Later it is proved that in Peano models it is possible to introduce recursive operations of any kind, but induction models where not all recursive operations can be introduced are also defined. The induction models which are not Peano models come in only two shapes: cycles and "spoons".

Chapter IV. Frames and General Structures.
This chapter consists of six sections. In the first, frames and general structures are informally introduced and the dichotomy between standard and non-standard views is discussed, arriving at the conclusion that it is intimately related with how the concept of subset is considered: that is, as a "logical" or undefined concept or as something to be defined in the logic.

In two other sections frames and general structures with the related semantics are introduced. The question of whether the general structures can be defined algebraically is dealt with in another section. The logics obtained by weakening comprehension, whose relevance is pointed out in this book, are also dealt with. The chapter closes by considering weak second order logic; that is, a logic where the concept of finiteness is taken from the metatheory and imposed as a "logical" concept.

Chapter V. Type Theory.
This chapter basically consists of the presentation of three different languages for type theory and a brief discussion of paradoxes and their solution in type theory. A deductive calculus for type theory is presented, which is a simple extension for all types of one for our second order calculus. The semantics of frames and general structures is also defined, and there is a very detailed proof of the equivalence of the usual definition of general structures with a proposed algebraic definition of general structures. The original functional presentation of Church is also treated, and the relationship with the previous relational presentation is given in full. Another section is devoted to equational type theory, a very illuminating logic where the only primitives are equality and abstraction and where the remaining logical concepts including connectors and quantifiers are definable. The chapter closes with our obtaining a calculus for this language. Limitations of time, space and knowledge have resulted in the omission of important subjects such as the beautiful definition of natural numbers in Church's paper, the connection of this presentation to the existing literature on models for typed lambda calculus, and perhaps also with Montague Grammar.

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Chapter VI. Many-sorted Logic.

This chapter consists of eight sections devoted to many-sorted logic. There is a long introduction where reasons for using many-sorted logic are provided. The fact that in many branches of mathematics and computer science the natural structures are many-sorted is highlighted. The standard treatment of reduction to single-sorted structures and logic is raised and the weak points of this reduction are considered. It is seen that many-sorted logic has been used with success in computer science and also in a wide range of logics, as a unifier logic. Many-sorted language, semantics and deductive calculus are presented in detail and the completeness of this calculus is proved in full by extending the usual method. The chapter closes with the reduction of many-sorted logic to first order unsorted logic in the classical way; namely, unification of domains for the conversion of structures and relativization of quantifiers in the translation of formulas.

Chapter VII. Applying Many-Sorted Logic.

In the final chapter, in line with the philosophy presented throughout the book, all the logics thus far discussed, including modal and dynamic logic, are translated into many-sorted logic and their usual structures are converted into many-sorted structures thus giving us many-sorted theories. These theories are the standard counterparts of the original logics and are capable of representing them faithfully. In addition, some of the metaproperties of many-sorted semantics and calculus are transferred to these logics. In the first section of this chapter the general plan of the translation, its aims and usual steps are discussed. The second section is devoted to the translation of higher order logic into many-sorted logic. Using this translation technique we obtain completeness and soundness results for higher order logic. Modal logic is afterwards introduced, and a many-sorted theory, **MODO**, is proposed which proves the usual semantic theorems of compactness, enumerability and Löwenheim-Skolem and also tests the calculus indirectly by proving soundness and completeness. **MODO**(S4) does the same for the modal logic S4. For **PML** the many-sorted theory proposed is $\mathbf{SOLO}^2$.

Readership.

This book is a monograph on extensions which can be used as an introductory texbook for Master's level students or senior undergraduates. It can also be used as a reference book since special attention has been given to the elaboration of conceptual distinctions and definitions. In fact, the book has been written like a novel, with a clear plot and an expected climax, and it is intended to be read as such, from cover to cover.

Since I wanted the book to be suitable for non-mathematicians too (including people not only in computer science, but also philosophers and linguists), it is not very demanding mathematically, the definitions are very detailed and the proofs are usually provided. Some choices were made in order to keep the book on a friendly level, but although the technical difficulty is rather low, a certain maturity of thinking is needed. Some of the frequent explanations attempt to achieve that in the form of a perspective of the logical landscape.

Prerequisites.

In order to read this book a modest knowledge of first order logic and set theory is needed, and it would be appropriate after introductory courses in both. For the benefit of potential readers short of the required background, some introductory books on first order logic and set theory are included in the bibliography.

Acknowledgments.

The book is also connected to my own intellectual and personal biography, not only in the obvious sense, in terms of the time it took me to write it, but because the subject has been around me (back and forth) for many years. The subject of my Master's thesis was completeness for second order logic and my PhD thesis was on second order logic as well. Both were presented in the Department of Logic in the University of Barcelona. It was decisive for this book that I spent the academic year 1977-1978 as a Fulbright Scholar in Berkeley, and that Leon Henkin guided me as my advisor. There I learnt many of the things that directly or indirectly, I hope, will show in these pages, including an intellectual appreciation of the beauty of teaching and the value of effort put into pedagogical issues. I have always thanked Leon Henkin for introducing me, with his enormous gifts for teaching, in a non-traumatic style, to his own wise overview of metamathematics and algebra. The subject presented was the usual one in graduate courses, but the presentation and insights were a challenge. The daily handwritten handouts in the unmistakable violet color of the Vietnamese copier... unforgettable!

When I first learnt about modal logic, the idea of translating it into first order many-sorted logic appeared immediately. Right from the start for me it was directly connected with what had been done in higher order logic. I have always had the feeling that this was just putting another piece of the puzzle in the right place. I soon discovered that this idea had already generated a whole industry and I became very happy afterwards when reading Johan van Benthem's survey and book. Wonderful, hats off! I applied the same treatment to dynamic logic and wrote a paper on this. I was in Leeds at that time, at the Centre for Theoretical Computer Science. Somehow this book grew from that paper and that visit.

In 1988 John Tucker and Karl Meinke organized an international workshop, held at the University of Leeds, where I met I Hajnal Andréka and Istvan Németi. Owing to what we like to call the Henkin connection, we shared the common ground that makes our understanding enjoyable. We talked about logic of programs, higher order logic, the general semantics of Henkin and the ontological, philosophical and practical consequences of the choice between standard or general semantics for it. I appreciate their help and encouragement during this period.

As this book begin to take shape and grow, several other people also helped me: Ildiko Sain, whose cleverness is only glimpsed in the incompleteness proof of Chapter II; Ramón Jansana, who suggested a shorter version of this incompleteness proof, which is included in the book. Several people, including Ramón and Johan van Benthem, questioned the advisability of including this incompleteness proof in such a book. I know they are right, and I apologize for my stubbornness, it has to do with the story of the proof and my emotional link with it until I obtained this readable, I feel, presentation. In addition, the computational importance of set-theoretic absoluteness can serve as a justification of my choice.

But, overall, I have to thank Antonia Huertas, whose support and help have been invaluable. Various places in the book show, I hope, some of her mathematical elegance. Of course, my students must be thanked, most especially Jordi López and Manuel Durán who suggested a few changes. I am also grateful to David Tranah for his patience and encouragement and to my Cambridge University Press referee for useful comments, on the whole.

In these pages of acknowledgments I also want to mention some of those whose indirect support helped me during this period; my sons and husband, my father, sister and brother, my female colleagues in the field of logic and philosophy of science and all my friends from Cervera del Maestre to San Diego.