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Theory of holors

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Theory of holors

A generalization of tensors

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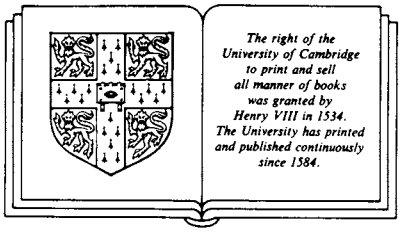

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To
DIRK J. STRUIK
our teacher and our friend

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Preface

The word *holor* indicates a mathematical entity that is made up of one or more independent quantities. Examples of holors are complex numbers, vectors, matrices, tensors, and other hypernumbers. The very fruitful idea of representing a collection of quantities by a single symbol apparently originated in 1673, when John Wallis represented a *complex number* as a point in 2-space. More complicated holors were developed by Hamilton (1843), Grassmann (1844), Gibbs (1881), Ricci (1884), Heaviside (1892), Levi-Civita (1901), Einstein (1916), and many others.

An unfortunate aspect of the subject is that each kind of holor was developed more or less independently, with its own nomenclature, its own theory, and its own textbooks. On the contrary, we propose to use a *single notation* that applies to all holors, be they tensors or nontensors. This step results in an immense simplification. Special symbols and notations, such as boldface type, ∇, grad, div, curl, are abandoned. Ordinary type is used, but with subscripts and superscripts. Such index notation is, of course, common in tensor theory but has not ordinarily supplanted the old notation of vector analysis and matrix theory. The first feature of our book, therefore, is the use of index notation throughout.

A second feature deals with coordinate transformation. A tensor is a holor that transforms in a particularly simple way. But many holors are not tensors. Indeed, applications can be cited where transformation has no physical significance. And even in applications where transformations are physically meaningful, the holor may not transform as a tensor. For instance, the ordinary derivative of a tensor is generally not a tensor.

Our book differs from most others in its generality. Not only does it cover nontensors having no transformation equation, but it also includes holors that transform in more complicated ways than ordinary tensors.

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This opens up an exciting field of new possibilities. In devising a holor for a new physical application, we are no longer limited to a few conventional types of holor but can tailor our system to fit the physics of the subject.

The authors look forward to the golden day when university curricula will not separate vector and tensor analysis into different subjects but will cover all holors in a uniform treatment. The present book constitutes a step in this direction. The material has been taught at MIT and at the University of Connecticut.

Finally, it is our pleasant duty to thank Dr. Struik for his kindness in reading the historical introduction and Cambridge University Press for its cheerful cooperation.

P.M.
D.E.S.

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Nomenclature

Nilvalent holor (Chapter 1): S.

Univalent holors (Chapter 1): $v^i = (v^1, v^2, \ldots, v^n)$, $v_i = (v_1, v_2, \ldots, v_n)$.

Bivalent holors (Chapter 1):

$$W^{ij} = \begin{bmatrix} W^{11} & W^{12} & \cdots & W^{1n} \\ W^{21} & W^{22} & \cdots & W^{2n} \\ \cdot & \cdot & \cdots & \cdot \\ W^{n1} & W^{n2} & \cdots & W^{nn} \end{bmatrix}, \qquad W^i_j = \begin{bmatrix} W^1_1 & W^1_2 & \cdots & W^1_n \\ W^2_1 & W^2_2 & \cdots & W^2_n \\ \cdot & \cdot & \cdots & \cdot \\ W^n_1 & W^n_2 & \cdots & W^n_n \end{bmatrix},$$

$$W_{ij} = \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1n} \\ W_{21} & W_{22} & \cdots & W_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ W_{n1} & W_{n2} & \cdots & W_{nn} \end{bmatrix}.$$

Trivalent holors (Chapter 1): W^{ijk}, W^{ij}_k, W^i_{jk}, W_{ijk}.

Symmetric part of holor u^{ij} (Chapter 1): $u^{(ij)} = \frac{1}{2}[u^{ij} + u^{ji}]$.

Antisymmetric part of holor u^{ij} (Chapter 1): $u^{[ij]} = \frac{1}{2}[u^{ij} - u^{ji}]$.

Kronecker delta (Chapter 1):

$$\delta^i_j = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

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Generalized Kronecker deltas (Chapter 1):

$$\delta^{ij}_{kl} = \begin{bmatrix} \delta^i_k & \delta^i_l \\ \delta^j_k & \delta^j_l \end{bmatrix}, \qquad \delta^{ijk}_{lmn} = \begin{bmatrix} \delta^i_l & \delta^i_m & \delta^i_n \\ \delta^j_l & \delta^j_m & \delta^j_n \\ \delta^k_l & \delta^k_m & \delta^k_n \end{bmatrix}.$$

Alternators (Chapter 1): δ^{12}_{ij}, δ^{ij}_{12}, δ^{123}_{ijk}, δ^{ijk}_{123}, ...

Determinant (Chapter 1): $|a^{ij}|$, $|a^i_j|$, $|a_{ij}|$.

Gamma products (Chapter 2): $\gamma^i_j v^j = w^i$, $w^{kl} = \gamma^{kl}_{ij} v^{ij}$, $\gamma^j_i u^i v_j = S$, $\gamma^{kj}_i u^i v_j = w^k$, $\gamma^{klj}_i u^i v_j = w^{kl}$, etc.

Scalar (Chapter 4): $S' = S$.

Contravariant univalent tensor (Chapter 4):

$$v^{i'} = \frac{\partial x^{i'}}{\partial x^i} v^i.$$

Covariant univalent tensor (Chapter 4):

$$u_{i'} = \frac{\partial x^i}{\partial x^{i'}} u_i.$$

Contravariant bivalent tensor (Chapter 4):

$$w^{i'j'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} w^{ij}.$$

Covariant bivalent tensor (Chapter 4):

$$w_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} w_{ij}.$$

Mixed bivalent tensor (Chapter 4):

$$w^{i'}_{j'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} w^i_j.$$

Contravariant univalent akinetor (Chapter 5):

$$\tilde{v}^{i'} = \sigma \frac{\partial x^{i'}}{\partial x^i} \tilde{v}^i.$$

Contravariant univalent akinetor of weight w (Chapter 5):

$$\tilde{v}^{i'} = \Delta^w \frac{\partial x^{i'}}{\partial x^i} \tilde{v}^i.$$

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Covariant univalent akinetor (Chapter 5):

$$\tilde{u}_{i'} = \sigma \frac{\partial x^i}{\partial x^{i'}} \tilde{u}_i.$$

Covariant univalent akinetor of weight w (Chapter 5):

$$\tilde{u}_{i'} = \Delta^w \frac{\partial x\, i}{\partial x\, i'} \tilde{u}_i.$$

Pseudoarea (Chapter 5): $d\tilde{\mathfrak{A}}_i = \delta^{123}_{ijk}\, dv^j\, dw^k$.

Pseudovolume (Chapter 5): $d\tilde{\mathfrak{V}} = \delta^{123}_{ijk}\, du^i\, dv^j\, dw^k$.

Gradient of a scalar (Chapters 4, 5):

$$\frac{\partial \phi}{\partial x^i}.$$

Pseudodivergence (Chapter 5):

$$\frac{\partial \tilde{v}^i}{\partial x^i}.$$

Pseudocurl (Chapter 5):

$$\tilde{W}^i = \delta^{ijk}_{123} \frac{\partial u_k}{\partial x^j}.$$

Pseudodivergence of a bivalent contravariant alternating akinetor of weight +1 (Chapter 5):

$$\frac{\partial \tilde{F}^{ij}}{\partial x^i}.$$

Linear connection (Chapter 7):

$$\Gamma^i_{jk} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k} \Gamma^{i'}_{j'k'} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}.$$

Torsion tensor (Chapter 7): $S^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj})$.

Covariant derivatives (Chapter 7):

$$\underset{1}{\nabla}_j v^k = \frac{\partial v^k}{\partial x^j} + v^l \Gamma^k_{jl}, \qquad \underset{2}{\nabla}_j v^k = \frac{\partial v^k}{\partial x^j} + v^l \Gamma^k_{lj}.$$

Riemann–Christoffel tensor of first kind (Chapter 8):

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^m_{jl} \Gamma^k_{im} - \Gamma^m_{il} \Gamma^k_{jm}.$$

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Riemann–Christoffel tensor of second kind (Chapter 8):

$$T^k_{ijl} = \frac{\partial \Gamma^k_{lj}}{\partial x^i} - \frac{\partial \Gamma^k_{li}}{\partial x^j} + \Gamma^m_{lj}\Gamma^k_{mi} - \Gamma^m_{li}\Gamma^k_{mj}.$$

Ricci tensors of first and second kind (Chapter 8):

$$R_{ij} = R^l_{ijl} = \frac{\partial \Gamma^l_{jl}}{\partial x^i} - \frac{\partial \Gamma^l_{il}}{\partial x^j}, \qquad T_{ij} = T^l_{ijl} = \frac{\partial \Gamma^l_{lj}}{\partial x^i} - \frac{\partial \Gamma^l_{li}}{\partial x^j}.$$

Ricci tensor of third kind (Chapter 10): $\mathfrak{R}_{ij} = R^l_{lij}$.

Absolute differentials (Chapter 9):

$$\underset{1}{\delta} v^i = dv^i + \Gamma^i_{jk} v^k\, dx^j, \qquad \underset{2}{\delta} v^i = dv^i + \Gamma^i_{kj} v^k\, dx^j$$

Riemannian metric (Chapter 10): $(ds)^2 = g_{ij}\, dx^i\, dx^j$.

Linear connection in metric space (Chapter 10):

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left[\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right].$$

Area (Chapter 10): $d\mathfrak{A}_i = g^{1/2}\delta^{123}_{ijk}\, dv^j\, dw^k$.

Volume (Chapter 10): $d\mathfrak{V} = g^{1/2}\delta^{123}_{ijk}\, du^i\, dv^j\, dw^k$.

Components of vector v^i (Chapter 10):

$$(v)_1 = \sqrt{g_{11}}\, v^1, \qquad (v)_2 = \sqrt{g_{22}}\, v^2, \qquad (v)_3 = \sqrt{g_{33}}\, v^3.$$

Magnitude of vector v^i (Chapter 10): $v = (g_{ij} v^i v^j)^{1/2}$.

Angle between vectors u^i and v^j (Chapter 10):

$$\cos\theta = \frac{g_{ij} u^i v^j}{uv}.$$

Scalar product of vectors (Chapter 10): $S = g_{ij} u^i v^j = \mathbf{u} \bullet \mathbf{v}$.

Vector product of vectors (Chapter 10):

$$w^k = g^{1/2} g^{kl} \delta^{123}_{lij} u^i v^j, \qquad \mathbf{w} = \mathbf{u} \times \mathbf{v}.$$

Gradient of scalar (Chapter 10):

$$\operatorname{grad}_i \phi = \frac{\partial \phi}{\partial x^i}, \qquad \operatorname{grad} \phi.$$

Divergence of vector (Chapter 10):

$$\operatorname{div} \mathbf{v} = g^{-1/2} \frac{\partial}{\partial x^i}(g^{1/2} v^i).$$

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Curl of vector (Chapter 10):

$$W^i = g^{-1/2}\delta^{ijk}_{123}\frac{\partial}{\partial x^j}(g_{kl}v^l), \qquad \text{curl } \mathbf{v}.$$

Scalar Laplacian (Chapter 10): $\nabla^2\phi = \text{div grad } \phi$.

Vector Laplacian (Chapter 10): $\nabla^2\mathbf{v} = \text{grad div } \mathbf{v} - \text{curl curl } \mathbf{v}$.

Euclidean metric (Chapter 11):

$$(ds)^2 = g_{ij}\,dx^i\,dx^j$$

where

$$g_{ij} = \sum_{i'=1'}^{n'}\left(\frac{\partial x^{i'}}{\partial x^i}\right)^2.$$

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