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1 Background on classical hydrodynamics

1.1 Equations of motion, potential flow and vorticity

Vortices, vortex motions and vortex interactions have been an important and vigorous branch of fluid mechanics since the early seventies. One reason is the belief that coherent structures exist in many types of turbulent flow and that the representation of turbulence as a superposition of interacting vortices may be useful. While the main subject of this book, quantized vortices, is part of the story of recent progress, and moreover contains a study of a form of turbulence of interacting quantized vortices, there has been remarkably little cross-fertilization of ideas between studies of classical and quantum vortices. Much of the material we wish to study here owes its origin to ideas which come from classical fluid mechanics. We therefore begin our study with a brief overview of some of those ideas. In doing so, we must recognize that the classical hydrodynamical ideas we shall need to refer to come from a wide variety of often rather specialized topics in fluid mechanics.

The equation of conservation of mass for a fluid of density ρ and velocity $\mathbf{v}$ is

$$\partial \rho / \partial t + \operatorname{div} \rho \mathbf{v} = 0 \tag{1.1}$$

In nearly all the discussion in this book, the fluids may be regarded as incompressible, and hence

$$\operatorname{div} \mathbf{v} = 0 \tag{1.2}$$

The equation of motion for an incompressible viscous fluid, the Navier–Stokes equation, is

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{v} = -\frac{1}{\rho} \operatorname{grad} p + \nu \nabla^2 \mathbf{v} \tag{1.3}$$

where $\mathbf{v}$ is the velocity, p the pressure and $\nu = \eta/\rho$ is the kinematic viscosity, the ratio of the dynamic viscosity η to density ρ. For a derivation

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of (1.3), and in particular the significance of the substantive derivative $d\mathbf{v}/dt$, one should consult a basic hydrodynamics text such as Landau and Lifshitz (1987) Section 2. The Navier–Stokes equation is usually very difficult to solve, even numerically. In special cases such as where boundaries are absent, there is often much to recommend the study of the flow of an inviscid, incompressible, adiabatic fluid, governed by Euler's equation

$$\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\text{grad})\mathbf{v}=-\frac{1}{\rho}\,\text{grad}\,p \qquad (1.4)$$

Equation (1.4) represents an ideal fluid in the sense that we are omitting internal frictional processes and hence viscosity and thermal conductivity. This ideal fluid motion is adiabatic and in many cases the entropy per gram S can simply be taken as a constant everywhere in the fluid. The flow is then described as *isentropic*. If w is the enthalpy per gram of the fluid, the relation $dw = T dS + V dp$, where T is the absolute temperature and V $(=1/\rho)$ the specific volume, reduces to $dw = V dp = dp/\rho$. We shall discover that liquid helium at absolute zero has many similarities to our ideal fluid. One important difference, however, is that the chemical potential per gram, defined as $d\mu = -S dT + V dp$ appears in place of the enthalpy.

We can then write Euler's equation in the form

$$\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla w \qquad (1.5)$$

and because of the vector identity

$$\tfrac{1}{2}\nabla v^2=\mathbf{v}\times(\nabla\times\mathbf{v})+(\mathbf{v}\cdot\nabla)\mathbf{v} \qquad (1.6)$$

a further version is

$$\partial\mathbf{v}/\partial t-\mathbf{v}\times(\nabla\times\mathbf{v})=-\nabla(w+\tfrac{1}{2}v^2) \qquad (1.7)$$

The solution of (1.4) for many flows is part of the background of classical mathematical physics. In some cases there are elegant exact solutions, but to many students it may seem that the ideal fluid is a sterile concept, especially compared to the demands of modern engineering such as aeronautics.

Hydrodynamicists, however, have long understood that classical mathematical fluid mechanics and practical engineering fluid mechanics can be combined by the concept of the boundary layer. That is, at high Reynolds numbers the flow near a boundary is described by (1.3) and that the remainder of the flow can be described by (1.4) with appropriate joining of solutions (we shall mention the boundary layer again shortly).

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One can then proceed to include in the flow of the 'inviscid' or 'ideal' fluid such features as circulation and vortices, which dominate the flow and whose generation depends on viscosity as we shall see below. In many cases the effects of viscosity on the evolution and interaction of vortices appear to be relatively slow, making the choice of an inviscid fluid for the discussion of vortices a sensible one.

The curl of the velocity field,

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{1.8}$$

is known as the vorticity and may be thought of as the circulation per unit area. We shall discuss circulation presently.

A curve drawn from point to point in the fluid, so that its direction is always that of the instantaneous direction of $\boldsymbol{\omega}$ is called a *vortex line.* The differential equation of the line comes from expressing the condition $\boldsymbol{\omega} \times \mathrm{d}\mathbf{l} = 0$ in components where $\mathrm{d}\mathbf{l}$ is taken along the vortex line. Because of the definition (1.8) we have immediately

$$\operatorname{div} \boldsymbol{\omega} = 0 \tag{1.9}$$

Equation (1.9) shows that vorticity behaves analogously to the velocity field of an incompressible fluid. If we draw a small closed curve C in the fluid and include every vortex line passing through this curve, we have obtained a *vortex tube.* It follows from Gauss' theorem that the integral of the normal component of $\boldsymbol{\omega}$ over any closed surface is zero:

$$\int_S \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S} = 0 \qquad \text{(any closed surface } S\text{)} \tag{1.10}$$

and if we apply this result to any two cross-sections $\mathrm{d}\mathbf{S}_1$ and $\mathrm{d}\mathbf{S}_2$ of the tube, the sides do not contribute and

$$\int_{S_1} \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S}_1 = \int_{S_2} \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S}_2 \tag{1.11}$$

Thus the flux of vorticity across any section of a vortex tube is conserved and represents a characteristic of the tube. Vortex tubes cannot terminate in the fluid, they must be closed or terminate on boundaries. Equations (1.9)–(1.11) are expressions of Helmholtz's first theorem.

The Navier–Stokes equation can be written in terms of the vorticity by taking the curl of (1.3):

$$\begin{aligned} \partial\boldsymbol{\omega}/\partial t &= \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} \\ &= (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} + \omega \nabla^2 \boldsymbol{\omega} \end{aligned} \tag{1.12a}$$

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Introducing a characteristic length L and velocity V we can make (1.12a) dimensionless through $\omega \to \omega V/L$, $t \to tL/V$ and $v \to vV$, yielding

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} + \frac{1}{Re}\nabla^2 \boldsymbol{\omega} \tag{1.12b}$$

with the Reynolds number $Re = VL/\nu$. Two geometrically similar viscous flows are dynamically similar at the same Reynolds number. For an ideal fluid, (1.12a) simplifies to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = 0 \tag{1.13a}$$

which leads to Helmholtz's second theorem. Consider a surface S enclosed by a contour C. Let $\mathrm{d}\mathbf{S}$ be an element of this surface. Multiplying scalarly by $\mathrm{d}\mathbf{S}$ and integrating over S, we obtain

$$\int_S (\partial \boldsymbol{\omega}/\partial t) \cdot \mathrm{d}\mathbf{S} - \int_S \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \cdot \mathrm{d}\mathbf{S} = 0 \tag{1.13b}$$

Transforming the second integral by Stokes' theorem, we have

$$\int_S (\partial \boldsymbol{\omega}/\partial t) \cdot \mathrm{d}\mathbf{S} + \int_C \boldsymbol{\omega} \cdot (\mathbf{v} \times \mathrm{d}\mathbf{l}) = 0 \tag{1.14}$$

where $\mathrm{d}\mathbf{l}$ is an element along the contour C defining the vortex tube at a particular cross-section. A careful argument (see Chandrasekhar (1961) Section 20) now shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S} = 0 \tag{1.15}$$

and that the integral of the normal component of $\boldsymbol{\omega}$ over any surface S bound by a closed curve, remains constant as we follow the surface S with the motion of the fluid elements constituting it:

$$\int_S \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S} = \text{constant} \tag{1.16}$$

Another statement is that the strength of a vortex tube is an integral of the equations of motion. We see that the vorticity $\boldsymbol{\omega}$ may be changed by 'stretching' the vortex tube, so long as the quantity $\boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S}$ remains the same.

Transforming (1.16) by Stokes' theorem we can define the circulation

$$\Gamma = \int_C \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_S \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S} = \text{constant} \tag{1.17}$$

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This definition shows why the vorticity may be thought of as the circulation per unit area. Equation (1.17) can be put in the form

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int \mathbf{v}\cdot d\mathbf{l} = 0 \tag{1.18}$$

or

$$\int \mathbf{v}\cdot d\mathbf{l} = \text{constant} \tag{1.19}$$

which is known as Kelvin's theorem, or the law of conservation of circulation.

Streamlines in the flow are made up of lines tangent to $\mathbf{v}$ at all points. Their differential equation comes from the relationship $d\mathbf{l}\times\mathbf{v}=0$, that is $dx/v_x = dy/v_y = dz/v_z$.

In the particular case of irrotational flow,

$$\boldsymbol{\omega} = 0 \qquad \text{(potential flow)} \tag{1.20}$$

and the circulation around any closed contour is zero:

$$\oint \mathbf{v}\cdot d\mathbf{l} = 0 \tag{1.21}$$

Thus closed streamlines cannot exist in potential flow unless the space involved is multiply-connected. In potential flow in such a region the circulation can be finite if the closed contour around which it is taken cannot be contracted to a point without crossing the boundaries of the region. This exception will prove to be all-important for our studies of helium II.

A particularly simple situation arises for potential flow of an incompressible fluid. For if $\nabla\times\mathbf{v}=0$ and $\nabla\cdot\mathbf{v}=0$, with the substitution

$$\mathbf{v} = \text{grad}\,\varphi \tag{1.22}$$

we obtain Laplace's equation

$$\nabla^2\varphi = 0 \tag{1.23}$$

for the velocity potential φ.

With Euler's equation in the form (1.5) we can substitute $\mathbf{v}=\nabla\varphi$, and obtain

$$\nabla(\partial\varphi/\partial t + \tfrac{1}{2}v^2 + w) = 0 \tag{1.24}$$

which yields the first integral

$$\partial\varphi/\partial t + \tfrac{1}{2}v^2 + w = f(t) \tag{1.25}$$

where $f(t)$ may be taken to be zero without dynamical consequences because the velocity is the space derivative of φ and we can replace φ

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by $\varphi + \int f(t)\,dt$ making the right-hand side of (1.25) zero. We see that in steady incompressible flow the greatest pressure occurs at the points where the velocity is least. For an ideal incompressible fluid in steady potential flow

$$\tfrac{1}{2}v^2 + w = \tfrac{1}{2}v^2 + p/\rho = \text{constant} \tag{1.26}$$

throughout the fluid, a statement referred to as Bernoulli's principle. More generally the constant in (1.26) takes different values along different streamlines, even if the flow is rotational. It can also be shown (Landau and Lifshitz, 1959, Section 6) that the quantity

$$\rho\mathbf{v}(\tfrac{1}{2}v^2 + w) \tag{1.27}$$

is the energy flux density vector for an ideal isothermal fluid. The magnitude of this quantity is the amount of energy passing through a unit area perpendicular to $\mathbf{v}$ in one second.

A well-known result of steady potential flow about an object is that it exerts no lift or drag on that object (d'Alembert's paradox). This is because (1.26) does not contain time explicitly and potential flow around a body, for example, depends only on the velocity and not on acceleration. On the other hand, if the flow is unsteady (i.e., the body is accelerating) φ must be taken into account and the body acts as if it had an effective mass equal to its physical mass plus some fraction of the mass of the liquid displaced by the body. For spheres, the fraction is $\frac{1}{2}$, for cylinders (per unit length) it is 1 (Landau and Lifshitz, 1959, Section 11).

A special situation arises if we consider potential flow of a fluid which has circulation Γ about a cylinder combined with a uniform steady flow v_∞ at large distances from the cylinder. By integrating the pressure over the cylinder one can show that while d'Alembert's paradox still holds (i.e., there is no drag on the cylinder), a lift force of magnitude $\rho v_\infty \Gamma$ per unit length is developed (the Kutta–Joukowski theorem). In viscous fluids the circulation can be produced by rotation of the cylinder in a fluid stream. This phenomenon was first observed by the German scientist Magnus in 1852 and is often referred to as the Magnus effect. Elementary reasoning from Bernoulli's principle gives the direction of the lift force, which can be put in vector form as

$$f_M = \rho\mathbf{v}_\infty \times \mathbf{\Gamma} \tag{1.28a}$$

This result is independent of the shape of the body. In the liquid helium literature this result is applied to vortices (Section 3.1) and is known as the Magnus force.

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The analogous result in electromagnetic theory is the Lorentz force $\mathbf{df}_L$ on a current element $i\mathrm{d}\mathbf{l}$ (in Gaussian units)

$$\mathbf{df}_L = (i/c)(\mathrm{d}\mathbf{l} \times \mathbf{B}) \tag{1.28b}$$

where i is the current, c the velocity of light and $\mathbf{B}$ the magnetic field. Applied to a superconducting wire (no field penetration) this gives a Lorentz force independent of the detailed shape of the conductor.

Fluid motion may be represented as a combination of translation, rotation and deformation. Vorticity is a measure of rotation. We show in Figure 1.1 the behavior of a square fluid element in rotation and distortion, as in a parallel shear flow between two moving planes. The component of vorticity is obtained by considering an infinitesimal plane element normal to this component: its vorticity is twice the average angular velocity of the lines, and if this average is nonzero, the fluid particle is changing its direction in space, i.e., it is rotating. The factor of 2 here is not a mystery. Tritton (1982) gives a useful discussion of vorticity. The circulation per unit area of a fluid in uniform rotation in a circle is $(\Omega R)(2\pi R)/\pi R^2 = 2\Omega$, expressing the fact that the product of the radius and circumference of a circle is twice its area. It is important

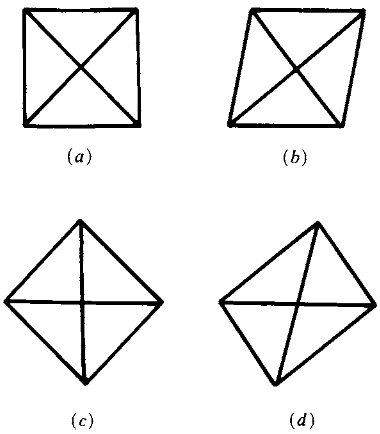

Figure 1.1 This example shows a square fluid element in rotation and distortion as in the simple shear flow of Figure 1.2(*a*). The change from (*a*) to (*b*) involves rotation of the vertical sides, but not the horizontal. There is thus nonzero average rotation and hence vorticity. The two diagonals have both rotated, but each only half as much (in the limit of small changes) as a vertical side. Diagrams (*c*) and (*d*) show the same process, but with the sides and diagonals interchanged (after Tritton (1982)).

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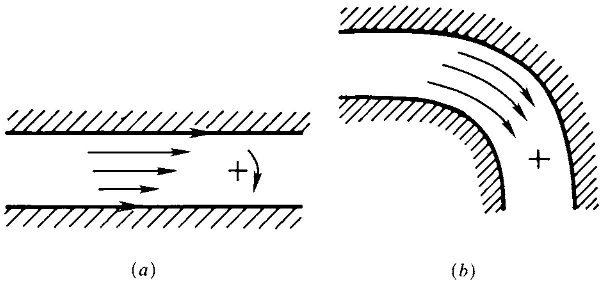

Figure 1.2 (*a*) A parallel shear flow (with the lower plane at rest and the upper plane in motion) has vorticity whereas (*b*) potential flow about a corner does not. In (*a*) a pair of diagonals will rotate as shown in Figure 1.1, while in (*b*) the orientation of diagonals in space does not change (after McCormack and Crane (1973)).

to note that there is a distinction between particle rotation, as indicated by vorticity, and motion along a curved path, as shown in Figure 1.2. One can have irrotational flow with curved particle paths and rotational flow with straight ones.

1.2 Creation of vorticity in classical viscous flows

The origin of vorticity lies in the effects of viscosity. The boundary condition for a viscous fluid is that it is at rest relative to a solid boundary. With motion, shear is produced at the boundary and, as we have noted, shear has vorticity. Vorticity generated at a boundary spreads into the bulk of the fluid by means of diffusion and convection. Vorticity transport is somewhat analogous to heat conduction and convection. The heat equation (Serrin, 1959) for a viscous incompressible fluid is

$$\rho C[\partial T/\partial t+(\mathbf{v}\cdot\nabla)T]=k\nabla^2 T+D \tag{1.29}$$

where k is the thermal conductivity, C the specific heat, T the absolute temperature and where the dissipation function

$$D=\eta[\boldsymbol{\omega}^2-2\operatorname{div}(\mathbf{v}\times\boldsymbol{\omega})+\nabla^2\mathbf{v}^2] \tag{1.30}$$

Equation (1.12) for vorticity transport simplifies for two-dimensional flows in the (x, y)-plane to

$$\partial\omega/\partial t+(\mathbf{v}\cdot\nabla)\omega=\nu\nabla^2\omega \tag{1.31}$$

where $\omega=\omega_z$ is the vorticity component normal to the (x, y)-plane. Equation (1.31) is analogous to the heat equation for vanishing dissipation D. The operation $(\mathbf{v}\cdot\nabla)$ represents convection and ∇^2 represents diffusion.

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Convection of vorticity has the property that vorticity is conserved on a particle path. Thus vorticity can be transferred to neighboring paths only by diffusion, i.e., by the effect of viscosity. At a boundary where the fluid is at rest, vorticity can be transferred only by diffusion.

A particularly simple example of generation of vorticity at a boundary occurs in the flow near an oscillating flat plate (Stokes' second problem). Consider a flat plate lying in the (x, z)-plane with a semi-infinite layer of fluid above it extending in the positive y-direction. If x represents the coordinate parallel to the direction of oscillation and y the coordinate perpendicular to the wall, the fluid must stick to the wall and

$$u(0, t) = U_0 \cos \omega t \qquad y = 0 \tag{1.32}$$

The fluid velocity $u(y, t)$ is the solution to the reduced Navier–Stokes equation

$$\partial u/\partial t = \nu(\partial^2 u/\partial y^2) \tag{1.33}$$

(the heat conduction equation). The solution is

$$u(y, t) = U_0 \exp(-ky) \cos(\omega t - ky) \tag{1.34}$$

where

$$k = (\omega/2v)^{1/2} \tag{1.35}$$

In this solution, two fluid layers a distance $2\pi/k = 2\pi(2\nu/\omega)^{1/2}$ apart oscillate in phase. Thus a distance k^{-1} is sometimes called the 'penetration depth' of the viscous wave. It is analogous to the temperature distribution of the earth's surface subject to diurnal heating from the sun and to the penetration of electromagnetic radiation into a conductor (skin effect).

The detailed way vorticity enters the fluid is complicated and geometry-dependent. One example is shown in Figure 1.3 where streamlines and equivorticity lines are shown for flow about a sphere at different Reynolds numbers $Re = vD/\nu$. For $Re \to 0$, the familiar Stokes solution is shown in Figure 1.3(a). There is fore–aft symmetry to the flow. But at $Re = 5$ we see this symmetry is broken (Figure 1.3(b)) and, even more so at $Re = 40$ (Figure 1.3(c)). By $Re = 100$ a vortex ring forms behind the sphere, as shown in the temporal series of flows in Figure 1.3(d).

Flow near a flat plate at high Reynolds numbers can be divided into two regions, a thin boundary layer where vorticity exists, and the region outside which can be considered inviscid. The concept is illustrated in Figure 1.4 which shows the velocity and vorticity distributions in a laminar boundary layer.

Boundary layers can detach from the wall at which they are formed. Such a tendency for a sphere is shown in Figure 1.3. There are then,

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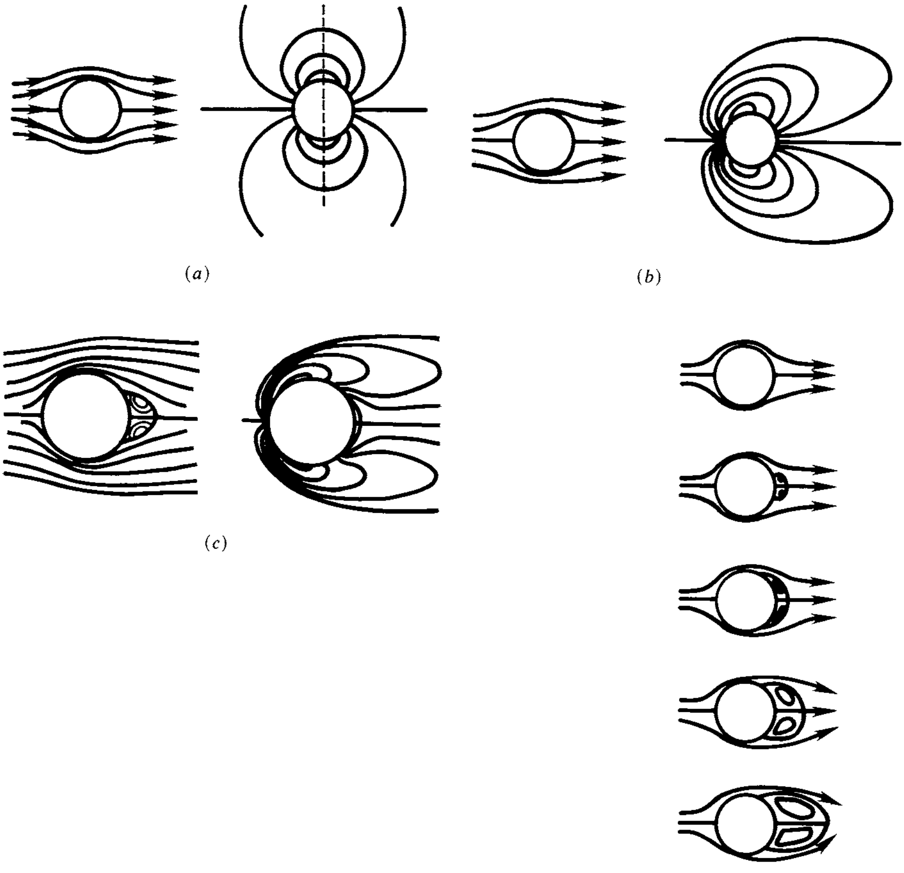

(*a*) (*b*) (*c*) (*d*)

Figure 1.3 Showing the flow past a sphere of a viscous incompressible fluid at various Reynolds numbers. The tendency to form a wake behind the sphere is evident. Diagrams (*a*) and (*b*) show streamline and equivorticity lines at $Re = 0$ and 5 (after Jenson (1959)). Diagram (*c*) shows streamlines and equivorticity lines at $Re = 40$ (after Dennis and Walker (1971)). Diagram (*d*) shows the temporal development of streamlines about a sphere at $Re = 100$. The formation of a vortex ring is clearly illustrated (after Rimon and Cheng (1969)).

within the fluid, regions where the velocity varies spatially. Note that the shear flows of Figure 1.4(*b*) differ only by a constant parallel stream, and hence have the same vorticity distribution. This demonstrates that the vorticity field is invariant to changes in inertial frame, unlike velocity.

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