Elementary Geometry of Differentiable Curves: an Undergraduate Introduction

C. G. Gibson

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2001

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13pt. *System* LaTeX 2_ε [TYPESET BY THE AUTHOR]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

ISBN 0 521 80453 1 hardback
ISBN 0 521 01107 8 paperback

To Dorle

Contents

List of Illustrations

Cambridge University Press
978-0-521-01107-5 - Elementary Geometry of Differentiable Curves: An Undergraduate Introduction
C. G. Gibson
Frontmatter
More information

List of Tables

Preface

For around three decades one of the distinguishing features of my department† has been a second year course on the geometry of curves, in which (following an earlier set of precepts) plane curves are studied simultaneously from the algebraic and differentiable viewpoints. It has a proven history of success, providing students with a wonderful introduction to visually attractive geometry. My experience in teaching this course convinced me that the time was ripe to raise the profile of undergraduate geometry. The algebraic viewpoint developed into my text, 'Elementary Geometry of Algebraic Curves' (Cambridge University Press, 1998). The present text is intended as a companion volume, representing the differentiable viewpoint.

0.1 Differential Geometry

Differential geometry is a fascinating area of mathematics, of substantial and increasing importance in the physical sciences. Despite that, the subject has a low billing in most undergraduate curricula, either not appearing or relegated to a final year optional course. That is a shame since much can be achieved with minimal mathematical preparation in the second year by restricting attention to plane curves: those who then wish to develop their interest can proceed to final year courses studying more general objects. Plane curves live in an environment familiar from school mathematics (the Euclidean plane), and have features readily visible on a computer screen: moreover, their study uses foundational mathematics (calculus, linear algebra and complex numbers) in a useful way, with only a handful of results using basic facts in real analysis. Most

† The Department of Mathematical Sciences in the University of Liverpool

computations use no more than elementary symbolic manipulation and differentiation, thus reinforcing the basic skills acquired in school and the first year of undergraduate study. The restriction to the plane has the further benefit that its natural complex structure can be profitably exploited to simplify both theory and practice.

0.2 Special Curves

One of my objectives was to ensure that the reader becomes increasingly familiar with a small zoo of special curves, mostly drawn from physical applications, to provide a framework on which the ideas of the subject can be hung. Many of the curves of historical significance can be studied profitably from either the algebraic or the differentiable viewpoint, so provide a useful starting point. Indeed it is healthy to bear in mind that the study of such special curves provided the genesis for the existing body of mathematics. Moreover some of these curves exhibit subtle features, requiring careful analysis. I resisted the temptation to expand upon the historical aspects: it would require a separate volume, and an author with much wider knowledge. However, most of these named curves are either of historical, or of mathematical significance – sometimes both. For instance the catenary was discovered by Galileo (who confused it with the parabola) and later studied by Jacques Bernoulli the Elder (who discovered its true form). But it is also of mathematical significance, as a plane section of the minimal surface spanning two circular discs, the only minimal surface of revolution.

0.3 Curvature, Contact and Caustics

Curvature is of course one of the central concepts. Here is one of the big ideas of mathematics, compelling in its simplicity, yet surprisingly subtle. In time honoured fashion it is shown that a curve is completely determined (up to congruence) by its speed and curvature functions. This seems to me to be an excellent illustration of Klein's 'Erlanger Programm', one of the few accessible to undergraduates prior to their final year of study. I wanted to ensure that students also had the opportunity of viewing elementary differential geometry from the (less familiar) singular viewpoint. Thus two chapters are devoted to studying the contact of curves with lines and circles, leading to an understanding of exceptional points on curves such as cusps, inflexions and vertices. The singular theme is continued via a discussion of envelopes. As a serious application, two chapters are devoted to caustics by reflexion, an attractive area of potential application to several areas of the physical sciences, which deserves to be better known. Caustics are already of considerable importance in geometric optics. However, their significance in acoustics is not so well established, and there is little doubt that they play a key role in understanding the mysteries of radio propagation.

0.4 Planar Kinematics

The later chapters develop a subject close to my heart, namely kinematics. Like my engineering colleagues, I regret the fact that this area of study has largely disappeared from mathematics degrees. It deserves putting into a historical context, for the subject is as old as mathematics itself. It relates intimately to two of the great social revolutions – the Power Revolution, when man was gradually released from the drudgery of providing a source of power (as ways became available of converting natural sources of power into mechanical work) and the subsequent Industrial Revolution. Planar motions with a single degree of freedom represent the core material of classical kinematics: it stands on its own as interesing geometry, having intimate and fruitful relations with other areas of mathematics. The simplest examples arise from the roulettes of curve theory, indeed it is one of the key results of the subject that general planar motions arise in this way. (It is for that reason that trochoids, curves traced by a point carried by one circle rolling on another, are introduced at an early stage.) Here again the singular viewpoint is of historical importance, producing the circle of inflexions and the cubic of stationary curvature, sadly far better known to engineers than mathematicians. And, looking to the future, one sees this classical core as the starting point of wider theoretical investigations into robotics, a subject which plays an ever increasing role in our everday lives.

0.5 Concerning the Structure

In keeping with my objective of writing for undergraduates, with a year of foundational mathematics behind them, this volume is unashamedly example based. It is my firm personal belief that there is much educational merit in really getting to grips with a range of explicit examples. The subject is indeed rich in attractive examples, many of which arose in the physical sciences, and are of historical significance. I believe that students gain in confidence from this approach, and enjoy the security of increasing familiarity with key examples. Those who wish to can always pursue abstract theory for its own sake by proceeding to higher degrees where their needs will be met. The format parallels that of the companion volume 'Elementary Geometry of Algebraic Curves'. I kept the individual chapters fairly short, on the theory that each chapter revolves around one new idea: likewise the sections are brief, and punctuated by a series of 'examples' illustrating the concepts. I included a substantial and coherent collection of exercises (many culled from the older literature) designed to illustrate (and even amplify) the small amount of theory. Each chapter contains sets of exercises, each appearing immediately after the relevant section.

0.6 Curve Tracing

A few words are in order on the subject of curve tracing. At the crudest level, there is a lot to be said for a thumbnail sketch of a curve to grasp the broadest qualitative features. Many interesting curves arise from simple geometric constructions, and can be traced on paper using no more than a school geometry set. The serious tracer will enjoy Lockwood's 'A Book of Curves' where such constructions are described in considerable detail. It is only one step further in this direction to acquire a spirograph, and enjoy the sheer beauty of the complex trochoids which it will trace. And if that does not satisfy you, seek out such gems as Alabone's gorgeous Edwardian collection† of colour illustrations produced by an intriguing mechanical device, the Epicycloidal Geometric Chuck! I mourn the decline in such scientific hobbies: however, those who feel that life is too short for such indulgences will find that computers provide superb renderings of curves in a fraction of a second. It is worth saying that curve tracing programs allow both student and teacher to experiment with curves interactively on a computer screen, thereby enhancing understanding of the underlying geometry. Most commercial mathematical software packages contain such programs: indeed many of the illustrations in this book were produced in MAPLE. The more adventurous will not find it difficult to write simple programs themselves illustrating individual curves, and even families of curves.

† 'Poly-cyclo-epicycloidal and other Geometric Curves'