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The Euclidean Plane

We need to lay firm foundations for later work. It is rather like gardening. We would like to have beautiful flowers appear immediately, but as gardeners well know, success is the result of careful preparation of the soil, and a generous measure of patience. So it is in mathematics. The soil in which our curves grow is the familiar plane of school geometry. Good preparation now will make our lives much easier at a later stage. Virtually everything in this book depends crucially on the 'Euclidean structure' of the plane. That provides the content of this introductory chapter, representing purely foundational material. For many readers this material will already be a part of their knowledge, in which case it might be best just to scan the contents and proceed to Chapter 2.

1.1 The Vector Structure

Throughout this text $\mathbb{R}$ will denote the set of real numbers. For linguistic variety we will sometimes refer to real numbers as *scalars*. We will work in the familiar real plane $\mathbb{R}^2$ of elementary geometry, whose elements $z = (x, y)$ are called *points* (or *vectors*). Recall that we can add vectors, and multiply them by scalars λ, according to the familiar rules

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ \lambda(x, y) &= (\lambda x, \lambda y). \end{aligned}$$

Two vectors $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are *linearly independent* when $x_1 y_2 - x_2 y_1 \neq 0$. Linear algebra tells us that if z_1, z_2 are linearly independent then they form a basis for $\mathbb{R}^2$, meaning that any vector z can be written uniquely in the form $z = \lambda_1 z_1 + \lambda_2 z_2$ for some scalars λ_1, λ_2: in that case we say that z has *coordinates* (λ_1, λ_2) relative to the basis z_1, z_2. The most familiar case is the *standard basis* comprising the vectors

$e_1 = (1,0)$, $e_2 = (0,1)$ giving rise to the *standard coordinates* of elementary geometry.

1.2 The Scalar Product

The plane is endowed with its standard *Euclidean structure*. By this we mean that for any two vectors $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ we have the standard *scalar product* (or *dot product*) defined by the relation

$$z_1 \bullet z_2 = x_1 y_1 + x_2 y_2.$$

The basic properties (Exercise 1.2.1) of the scalar product are listed below.

S1: $z \bullet z \geq 0$ with equality if and only if $z = 0$.
S2: $z \bullet w = w \bullet z$.
S3: $z \bullet (\lambda w) = \lambda (z \bullet w)$.
S4: $z \bullet (w + w') = z \bullet w + z \bullet w'$.

S2 is referred to as the *symmetry* property. Properties S3, S4 together say that $\bullet$ is linear in its second argument: by symmetry, it is also linear in its first argument, and for that reason $\bullet$ is said to be *bilinear*. Two vectors z, w are *orthogonal* when $z \bullet w = 0$.

Example 1.1 Let z, w be vectors with $w \neq 0$. We claim that there is a unique scalar λ with the property that the vectors $z' = z - \lambda w$, w are orthogonal. Indeed, our requirement is that

$$0 = z' \bullet w = (z - \lambda w) \bullet w = z \bullet w - \lambda (w \bullet w),$$

giving the unique solution $\lambda = z \bullet w / w \bullet w$. We call the vector λw the *component of z parallel to w*, and the vector $z' = z - \lambda w$ the *component of z orthogonal to w*.

Exercises

1.2.1 Starting from the definition of the scalar product, establish the properties S1, S2, S3, S4.

1.3 Length, Distance and Angle

Property S1 of the scalar product is expressed by saying that the scalar product is *positive definite*. In view of this property it makes sense to

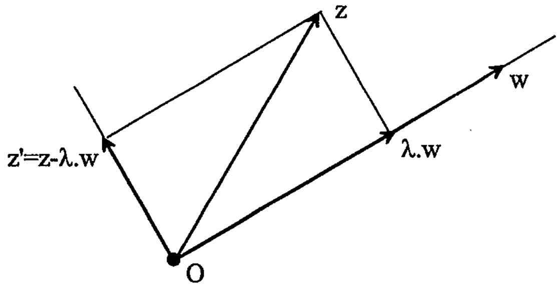

Figure 1.1. Components of a vector

define the *length* of a vector $z = (x, y)$ to be

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z \bullet z}.$$

Throughout this book we will use the following fundamental properties L1, L2, L3 of the length function. The property L1 is an immediate consequence of S1 above: however, L2 and L3 require proof, representing the next step in our development.

L1:	$\lvert z\rvert = 0$ if and only if $z = 0$.	
L2:	$\lvert z \bullet w\rvert \leq \lvert z\rvert\lvert w\rvert$.	(The Cauchy Inequality.)
L3:	$\lvert z + w\rvert \leq \lvert z\rvert + \lvert w\rvert$.	(The Triangle Inequality.)

Lemma 1.1 *For any two vectors z, w in the plane we have the relation* $|z \bullet w| \leq |z||w|$. (The Cauchy Inequality.)

Proof When $z = 0$ then the LHS is zero, and the inequality is satisfied. We can therefore assume that $z \neq 0$, so $z \bullet z > 0$. Set $\lambda = z \bullet w / z \bullet z$. Then λw represents the component of z parallel to w, and $z - \lambda w$ is the component orthogonal to w. (Figure 1.1.) Then

$$\begin{aligned} 0 &\leq |w - \lambda z|^2 = (w - \lambda z) \bullet (w - \lambda z) \\ &= w \bullet w - 2\lambda(z \bullet w) + \lambda^2(z \bullet z) \\ &= w \bullet w - \lambda(z \bullet w) \\ &= |w|^2 - \frac{(z \bullet w)^2}{|z|^2}. \end{aligned}$$

The result follows on multiplying through by $|z|^2$ and taking positive square roots. □

The Cauchy Inequality can be rephrased by saying that for non-zero vectors z, w we have

$$-1 \leq \frac{z \bullet w}{|z||w|} \leq 1.$$

Looking at the graph of the cosine function we see that there is therefore a unique angle θ with $0 \leq \theta \leq \pi$ for which

$$\cos\theta = \frac{z \bullet w}{|z||w|}.$$

We call θ the *angle* between the vectors z, w. This provides a good intuition for the meaning of the scalar product. The vectors $z = (x, y)$ for which $|z| = 1$ are called *unit* vectors: they are the vectors which lie on the circle $x^2 + y^2 = 1$ of radius 1 centred at the origin. When z, w are unit vectors the scalar product is just the cosine of the angle between them. It is for this reason that we call two vectors z, w 'orthogonal' when $z \bullet w = 0$, since then the cosine is zero, and the angle is $\theta = \pi/2$.

Lemma 1.2 *For any two vectors z, w in the plane we have the relation* $|z + w| \leq |z| + |w|$. (The Triangle Inequality.)

Proof The Cauchy Inequality yields the following series of relations, from which the result follows on taking positive square roots:

$$\begin{aligned} |z+w|^2 &= (z+w) \bullet (z+w) = z \bullet z + 2(z \bullet w) + w \bullet w \\ &\leq |z|^2 + 2|z \bullet w| + |w|^2 \leq |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

□

We define the *distance* between two points u, v in the plane to be the scalar $|u - v|$. The following basic properties of the distance function are immediate from L1, L2, L3.

M1: $|u - v| = 0$ if and only if $u = v$.
M2: $|u - v| = |v - u|$.
M3: $|u - v| \leq |u - w| + |w - v|$.

Note that distance is *invariant under translation*, in the sense that for any vector w the distance between u, v equals that between their translates $u + w$, $v + w$.

Example 1.2 Let a, b, c be non-zero vectors with $c = a - b$, and let θ be the angle between a, b. Expanding the expression for $|c|^2$ we obtain the *cosine rule* of school trigonometry

$$|c|^2 = |a|^2 - 2|a||b|\cos\theta + |b|^2.$$

A special case arises when a, b are orthogonal, so the angle θ is a right angle: the cosine rule then reduces to the familiar *Pythagoras Theorem*

$$|c|^2 = |a|^2 + |b|^2.$$

A basis u, v for the Euclidean plane is *orthogonal* when u, v are orthogonal: and it is *orthonormal* when it is orthogonal, and u, v are in addition unit vectors.

Example 1.3 Let T, N be orthogonal unit vectors. (The symbols are chosen deliberately to reflect the geometric situations we will meet throughout this text.) Thus the assumptions are that $T \bullet N = 0$, $T \bullet T = 1$, $N \bullet N = 1$. Observe first that T, N are linearly independent. Suppose indeed that we had a relation $\tau T + \nu N = 0$ for some scalars τ, ν: then, taking scalar products of each side of this relation with T, N, we see that $\tau = 0$, $\nu = 0$. By linear algebra T, N form an orthonormal basis for the plane, so that any vector v can be written uniquely as a linear combination $v = \tau T + \nu N$ for some scalars τ, ν. These scalars are very easily determined. Taking scalar products with T we see that $\tau = v \bullet T$; and likewise, taking scalar products with N we see that $\nu = v \bullet N$. Thus the required linear combination is

$$v = (v \bullet T)T + (v \bullet N)N. \tag{1.1}$$

The most familiar example of an orthonormal basis is the *standard* basis $T = (1, 0)$, $N = (0, 1)$. However, in Chapter 5 we will see that orthonormal bases can be associated in a very natural way to general points on curves, and that the way in which they change as we move along the curve provides geometric information of fundamental importance.

Exercises

1.3.1 Let u, v be any vectors in the plane. Establish the *parallelogram law*

$$|u + v|^2 + |u - v|^2 = 2|u|^2 + 2|v|^2.$$

1.3.2 The length of a vector was expressed in terms of the scalar product. Conversely, show that the scalar product can be expressed in terms of the length via the *Polarization Identity*

$$u \bullet v = \frac{1}{2}\left\{|u|^2 + |v|^2 - |u-v|^2\right\}.$$

1.4 The Complex Structure

One of the gains of working throughout with *planar* curves is that we will be able to take advantage of the fact that the points $z = (x, y)$ in the plane can be identified with complex numbers $z = x + iy$. Under this identification, vector addition of points in the plane corresponds to the usual addition of complex numbers. The gain is largely one of notational and computational efficiency (rather than mathematical understanding) but is still worthwhile. We will adopt standard notations by writing $x = \Re z$ for the *real part* and $y = \Im z$ for the *imaginary part* of the complex number $z = x + iy$. Likewise, the *complex conjugate* is written $\overline{z} = x - iy$, identified with the reflexion $z = (x, -y)$ of the point $z = (x, y)$ in the x-axis. The real gain lies in the availability of multiplication and division for complex numbers. Recall that the *product* of two complex numbers $z = x + iy$, $w = u + iv$ is defined to be

$$zw = (xu - yv) + i(xv + yu).$$

In particular, for any complex number $z = x+iy$, identified with the point $z = (x, y)$, we have $iz = -y + ix$ identified with the point $iz = (-y, x)$ obtained by rotating $z = (x, y)$ anticlockwise about the origin through a right angle.

Example 1.4 Recall that the component of a vector a in the direction of a unit vector b is the vector $(a \bullet b)b$. (Example 1.1.) It is useful to express this in complex notation. Note that for *any* two vectors a, b we have $2(a \bullet b) = a\overline{b} + \overline{a}b$: in particular when b is a *unit* vector (i.e. $b\overline{b} = |b|^2 = 1$) we have $2(a \bullet b)b = a + \overline{a}b^2$.

Note that the length of the vector $z = (x, y)$ is the *modulus* $|z|$ of the corresponding complex number $= x + iy$: in particular, vectors of unit length correspond to complex numbers of unit modulus. Recall that any vector $z = (x, y)$ of unit length can be written $z = (\cos t, \sin t)$ for some real number t: under the identification with complex numbers this means that any unit complex number u can be written $u = e^{it}$ where by definition $e^{it} = \cos t + i \sin t$. More generally, any complex number $z \neq 0$ can be written uniquely in the polar form $z = re^{it}$ where $r = |z|$.

1.5 Lines

Lines will play a fundamental role in studying the geometry of general curves, so it is worth recalling some basic facts. We define a *line* to be the set L of points (x, y) satisfying an equation of the form $ax + by + c = 0$, where a, b, c are real numbers, and at least one of a, b is non-zero. In this section we will review the basic properties of lines via a series of examples. The first example expresses the most basic property of all, namely that the equation of a line is determined (up to scalar multiples) by any two of its points.

Example 1.5 Through any two distinct points $p = (p_1, p_2)$, $q = (q_1, q_2)$ in $\mathbb{R}^2$ there is a unique line $ax + by + c = 0$. We seek scalars a, b, c (not all zero) for which

$$ap_1 + bp_2 + c = 0, \qquad aq_1 + bq_2 + c = 0.$$

Since p, q are distinct, the 2×3 coefficient matrix of these two linear equations in a, b, c has rank 2. By linear algebra it has kernel rank 1, so there is a non-trivial solution (a, b, c), and any other solution is a non-zero scalar multiple of this one. Explicitly, the line joining p, q has the equation

$$(p_1 - q_1)(y - p_2) = (p_2 - q_2)(x - p_1).$$

What is important for our purposes is that lines can be 'parametrized' in a natural way, providing a model for the general 'parametrized' curves of Chapter 2.

Example 1.6 Consider a line $ax + by + c$, and *distinct* points $p = (p_1, p_2)$, $q = (q_1, q_2)$ on the line. Then a brief calculation verifies that for any choice of scalar t the point $r = (1 - t)p + tq$ also lies on the line. Conversely, we claim that any point $r = (r_1, r_2)$ on the line has the form $r = (1 - t)p + tq$ for some scalar t. The proof goes as follows. Since p, q, r all lie on the line we have

$$\begin{cases} ap_1 + bp_2 + c &= 0 \\ aq_1 + bq_2 + c &= 0 \\ ar_1 + br_2 + c &= 0. \end{cases}$$

That is a linear system of three equations in a, b, c. Since at least one of a, b is non-zero, the system has a non-trivial solution. By linear algebra, the 3×3 matrix of coefficients is singular, so the rows $(p_1, p_2, 1)$, $(q_1, q_2, 1)$, $(r_1, r_2, 1)$ are linearly dependent. However, the first two rows are linearly independent (as p, q are distinct) so the third row is a linear combination of the first two; thus there exist scalars s, t for which

$$(r_1, r_2, 1) = s(p_1, p_2, 1) + t(q_1, q_2, 1).$$

It follows immediately that $r = sp + tq$ and $1 = s + t$ from which we deduce the required relation $r = (1-t)p + tq$.

In view of this result we introduce the following definitions. Given two distinct points $p = (p_1, p_2)$, $q = (q_1, q_2)$ the *standard parametrized* line through p, q is the specific parametrization given by the formulas below, with the points p, q corresponding respectively to the parameters $t = 0$, $t = 1$.

$$x = (1-t)p_1 + tq_1, \qquad y = (1-t)p_2 + tq_2.$$

Example 1.7 The standard parametrization of a line depends on the choice of points p, q. For instance we can parametrize the line $y = 0$ via the points $p = (0,0)$, $q = (1,0)$ to obtain the parametrization $x = t$, $y = 0$: on the other hand the points $p = (-1,0)$, $q = (1,0)$ give rise to the parametrization $x = 2t - 1$, $y = 0$.

Example 1.8 The scalar product provides a convenient way of writing down the equation of a line. For any fixed non-zero vector $N = (a,b)$ the vectors $z = (x,y)$ orthogonal to N are those for which $ax + by = 0$. That is a single linear equation in the variables x, y satisfied by $x = -b$, $y = a$, so by linear algebra its solutions (x,y) lie on a line through the origin spanned by the vector $(-b,a)$. Thus the equation $ax + by = 0$ of any line through the origin can be written in the form $N \bullet z = 0$ for some non-zero vector N. More generally, the equation $ax + by + c = 0$ of *any* line can be written in the form $N \bullet z + c = 0$ for some non-zero vector N. Since the equation of a line is unique up to scalar multiples, the vector N likewise is unique up to scalar multiples: it is an example of a 'normal' vector. (Chapter 2.) In particular, in writing down the equation of a line we can always choose N to be a *unit* vector.

Example 1.9 Let N be a non-zero vector. Given a fixed point z_0, the equation of the line L through z_0 orthogonal to N can be written in the

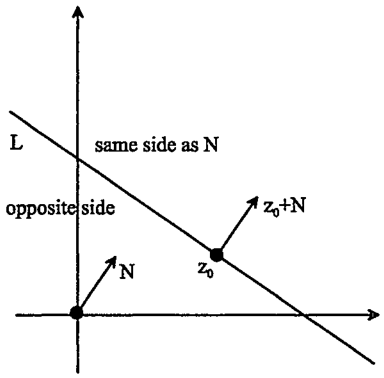

Figure 1.2. The two sides of a line

form $(z - z_0) \bullet N = 0$: indeed (as we saw in the previous example) that is the equation of a line orthogonal to N, and it certainly passes through z_0. For visualization purposes it helps to think of N with its origin at the point z_0: more precisely, we think in terms of $z_0 + N$ rather than N. Observe that the plane is partitioned into three sets by the relations

$$(z - z_0) \bullet N > 0, \quad (z - z_0) \bullet N = 0, \quad (z - z_0) \bullet N < 0$$

representing respectively one side of L, the line L itself and the other side of L. At a later stage in this book it will help us to be clear about which side is which. To this end, write θ for the angle between the vectors $z - z_0$ and N, so that

$$(z - z_0) \bullet N = \cos\theta |z - z_0||N|.$$

It follows immediately that the sign of $(z - z_0) \bullet N$ agrees with that of $\cos\theta$. Looking at the graph of the cosine function we see that the sign is positive if and only if $-\pi/2 < \theta < \pi/2$. One vector z on that side of L is the vector $z = z_0 + N$: for that reason we refer to the side of L with $(z - z_0) \bullet N > 0$ as being on the *same side* as N, and the other side as the *opposite side*.

Example 1.10 Let $N = (a, b)$ be a non-zero vector, and let L be a line with equation $z \bullet N = c$. By a *direction* for the line we mean any vector T orthogonal to N: it is an example of a 'tangent' vector. (Chapter 2.) Thus we could take $T = (-b, a)$. Alternatively, we could choose any two distinct points p, q on the line, and take $T = q - p$: the relation

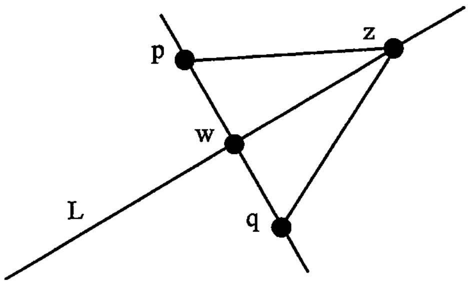

Figure 1.3. The orthogonal bisector

$T \bullet N = 0$ then follows from $p \bullet N + c = 0$, $q \bullet N + c = 0$ on subtraction. On this basis we say that two lines $a_1x + b_2y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ are *orthogonal* when the corresponding 'normal' vectors $N_1 = (a_1, b_1)$, $N_2 = (a_2, b_2)$ are orthogonal, or equivalently the corresponding 'tangent' vectors $T_1 = (-b_1, a_1)$, $T_2 = (-b_2, a_2)$ are orthogonal: either way, the condition is that $a_1a_2 + b_1b_2 = 0$.

Example 1.11 Let p, q be distinct points. A point z is *equidistant* from p, q when the distance from z to p equals the distance from z to q: the set of all points equidistant from p, q is called the *orthogonal bisector* of the line segment joining p, q. (Figure 1.3.) The orthogonal bisector is a line. Indeed the constraint on z is that $|p-z|^2 = |q-z|^2$: expanding both sides of this relation we obtain

$$2(q-p) \bullet z = |q|^2 - |p|^2$$

which is the equation of a line L, orthogonal to the vector $(q-p)$. Note that the *mid-point* $w = \frac{1}{2}(p+q)$ of the line segment joining p, q lies on the orthogonal bisector. (A formal definition of the term 'line segment' will be given in Chapter 2.)

Example 1.12 Using complex notation, consider two points $z = x + iy$, $w = u + iv$ equidistant from 0, and from 1. We claim that either $z = w$ or $z = \overline{w}$. The equidistance conditions are expressed by the relations

$$x^2 + y^2 = u^2 + v^2, \quad (x-1)^2 + y^2 = (u-1)^2 + v^2.$$

Subtracting these relation yields $u = x$, and hence $v = \pm y$: in the '+' case $w = z$, and in the '−' case $w = \overline{z}$.