1. SOUND WAVES

1.1 The wave equation

As remarked in the prologue, it is a balance between the compressibility and the inertia of a fluid that governs the propagation of sound waves through it. The linear theory of this propagation is described in chapter 1.

Use of a linear theory, for waves of any kind, implies that we consider disturbances so weak that in equations of motion we can view them as small quantities whose products are neglected. Such products of small quantities occur, for example, in the well-known expression for the acceleration of a fluid element:

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u}, \tag{1}$$

where $\mathbf{u}$ is the vector velocity field. In this expression (significant whenever inertia is important, as it is for practically all waves in fluids) the linear term $\partial \mathbf{u}/\partial t$ represents the local rate of change of $\mathbf{u}$ at a fixed point, while the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ describes how the element's velocity changes owing to its changing position in space. This 'convective rate of change' of $\mathbf{u}$ involves products of its spatial gradients with components of $\mathbf{u}$ itself, and so is neglected in a linear theory.

In this chapter, then, disturbances are supposed weak enough for such nonlinear contributions to inertial effects, together with nonlinear terms in the restoring forces (here, those associated with compressibility), to be neglected. Investigations of just *how* weak disturbances need to be for the theory to be reasonably good, and of what detailed effect on stronger disturbances the nonlinear terms may have, are postponed to chapter 2.

In this section, taking into account compressibility and inertia but no other properties of the fluid, we obtain the linearised equations of the theory of sound in their simplest form, a very useful one. We postpone consideration of how sound waves are influenced by effects neglected here (especially viscosity, heat conduction, external forces including gravity, and inhomogeneities such as stratification) to section 1.2 and later parts of the book.

The inertial nature of a fluid of density ρ is expressed when we apply to a small fluid element Newton's second law of motion. This demands that the product of the mass per unit volume ρ and of the acceleration (1) is the force on the element per unit volume, which in the absence of external forces is due solely to those internal stresses through which neighbouring fluid acts on it. When viscous stresses are neglected, this force per unit volume is simply minus the gradient ∇p of the fluid pressure p; thus

$$\rho(\partial \mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p. \tag{2}$$

Compressibility implies that the density of a fluid element may change, in accordance with the well-known equation of continuity:

$$\partial\rho/\partial t + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0. \tag{3}$$

The first two terms in (3) make up the total rate of change of ρ for the element. Thus, the *divergence* $\nabla\cdot\mathbf{u}$ of the velocity field is identified by (3) as the rate of increase of volume of an element moving in that velocity field, divided by the volume; in other words (since the element's mass is conserved) *minus* the rate of increase of density divided by the density. At the same time, an alternative interpretation of equation (3), based on grouping the second and third terms together as $\nabla\cdot(\rho\mathbf{u})$, is also possible and is used later (section 1.10).

We linearise these equations by regarding as small quantities all departures from a state in which the fluid has uniform density ρ_0 and is at rest. In the absence of external forces this implies that the pressure also takes a uniform value, say p_0.

Equations (2) and (3), with products of small quantities neglected, become the linearised equations of momentum

$$\rho_0\,\partial\mathbf{u}/\partial t = -\nabla p \tag{4}$$

and of continuity

$$\partial\rho/\partial t = -\rho_0\nabla\cdot\mathbf{u}. \tag{5}$$

These forms result from the neglect in (2) of $\mathbf{u}\cdot\nabla\mathbf{u}$ as already discussed, and the similar neglect in (3) of $\mathbf{u}\cdot\nabla\rho$, both involving products of small velocities with small gradients. At the same time the factor ρ in one term of each equation is replaced by ρ_0, the error being the product of a small quantity $(\rho-\rho_0)$ with another small quantity ($\partial\mathbf{u}/\partial t$ or $\nabla\cdot\mathbf{u}$). From this there result local rates of change of velocity $\mathbf{u}$ and density ρ directly proportional to pressure gradient and to velocity divergence, respectively.

One quantity that on the linear theory of sound behaves extremely simply is the vorticity; that is,

$$\boldsymbol{\Omega} = \nabla\times\mathbf{u}, \tag{6}$$

the curl of the velocity field (for general properties of vorticity, see texts on fluid dynamics). In fact, equation (4) implies that

$$\partial \mathbf{\Omega}/\partial t = 0, \tag{7}$$

since the curl of ∇p vanishes. Thus, the vorticity field is independent of time: vorticity 'stays put' on the approximations involved in the linear theory of sound, however much other quantities may be propagated.

This conclusion may astonish a reader familiar with Helmholtz's theorem that 'vortex lines move with the fluid'; all such changes due to convection are neglected in a linear theory. This is reasonable, however, in a theory of sound, which predicts that changes in other quantities (such as pressure) propagate at hundreds of metres per second, compared with which convection by relatively small flow velocities appears negligible.

That *rotational* part of the velocity field which is 'induced' by the vorticity field $\mathbf{\Omega}$ must, according to equation (7), be independent of time. The remaining part of the velocity field is irrotational and so can be written as the gradient $\nabla\phi$ of a 'velocity potential' ϕ. Only this part exhibits the fluctuations associated with sound propagation.

From this point on, we write

$$\mathbf{u} = \nabla\phi, \tag{8}$$

so that $\mathbf{u}$ is taken as the irrotational part of the velocity field (what is left after subtracting the velocities induced by the steady vorticity field). On linear theory, this irrotational propagating velocity field shows *no interaction* with any steady rotational flow field. Actual propagation of sound fields across vortex lines that actually move with the fluid is studied later (section 4.6); when flow velocities are very small compared with the sound speed the interaction is found to involve at most very gradual changes.

Equations (4) and (8) imply that

$$p - p_0 = -\rho_0\, \partial\phi/\partial t, \tag{9}$$

because the *gradients* of both sides of (9) are everywhere equal, and because both sides vanish in undisturbed parts of the fluid provided that the velocity potential, as is usual, is taken as the solution of (8) vanishing in such parts. Equation (9) differs from the well-known 'Bernoulli equation' for unsteady irrotational flows by omission of the term $-\frac{1}{2}\rho_0(\nabla\phi)^2$ (which on a linear theory is negligible) on the right-hand side.

Equations (5) and (8) express the rate of change of density as

$$\partial\rho/\partial t = -\rho_0 \nabla^2\phi \tag{10}$$

in terms of the Laplacian $\nabla^2\phi$, but no further progress beyond (9) and (10) can be made until the compressibility properties of the fluid have been used to infer an explicit relationship between changes of pressure and density. The character of such a relationship is discussed below (section 1.2) but here we simply assume a functional dependence $p = p(\rho)$. Linearising this means expanding in a Taylor series about $\rho = \rho_0$ as

$$p = p(\rho_0) + (\rho - \rho_0)p'(\rho_0) + \dots \tag{11}$$

and neglecting all terms beyond those shown as involving squares and higher powers of $\rho - \rho_0$. Then

$$\partial p/\partial t = p'(\rho_0)\,\partial\rho/\partial t, \tag{12}$$

whence we deduce, substituting for p on the left from (9) and for $\partial\rho/\partial t$ on the right from (10), that

$$\partial^2\phi/\partial t^2 = c^2\nabla^2\phi, \tag{13}$$

where the constant c (with the dimensions of velocity) is defined by the equation

$$c^2 = p'(\rho_0). \tag{14}$$

Most readers will recognise equation (13) as 'the wave equation': an equation characteristic of any phenomena, with energy conserved, involving propagation through a homogeneous medium at a single wave speed c, independent of waveform or direction of propagation. It is satisfied, for example, by components of electromagnetic fields in free space with c as the velocity of light $3 \times 10^8\,\mathrm{m\,s^{-1}}$. We find, however (section 1.2), that the sound speed c given by (14) is smaller by several orders of magnitude.

A simple solution of (13), representing a 'plane wave' travelling in the positive x-direction, is

$$\phi = f(x - ct). \tag{15}$$

Here $f(x)$ is the waveform at time $t = 0$, while the waveform at a later time t has identical shape but is shifted a distance ct in the positive x-direction. The wave is 'longitudinal' in the sense that the velocity field $\mathbf{u} = (u, v, w)$, satisfying

$$u = f'(x - ct), \quad v = w = 0, \tag{16}$$

is parallel to the direction of propagation.

For such a travelling plane wave, equation (9) gives

$$p - p_0 = \rho_0 c u, \tag{17}$$

a simple proportionality between excess pressure and the component of fluid velocity in the direction of propagation. This proportionality arises because, at a point in a travelling wave where the pressure is increasing, the gradient of pressure in the direction of propagation takes the negative value $-c^{-1}\partial p/\partial t$ and so *accelerates* the fluid, whose corresponding component of acceleration $\partial u/\partial t$ (see (4)) takes the positive value $(\rho_0 c)^{-1}\partial p/\partial t$. Note that (17) gives much bigger pressure variations for given u than would be obtained in steady flows through $\frac{1}{2}\rho_0 u^2$ terms.

Equation (15) is not the only solution of the wave equation which depends on just the two variables x and t; another such solution is

$$\phi = g(x+ct), \tag{18}$$

and the general solution is the sum of (15) and (18) with the functions f and g arbitrary. Equation (18) represents a plane wave travelling in the negative x-direction; the velocity field $\mathbf{u} = (u, v, w)$ satisfies

$$u = g'(x+ct), \quad v = w = 0 \tag{19}$$

and the excess pressure is
$$p - p_0 = -\rho_0 cu, \tag{20}$$

but of course the ratio of excess pressure to the velocity component in the direction of propagation $(-u)$ is still $+\rho_0 c$.

The general plane wave, travelling in the direction of the vector (ξ, η, ζ), is

$$\phi = h(\xi x + \eta y + \zeta z - ct), \tag{21}$$

which satisfies (13) provided that $\xi^2 + \eta^2 + \zeta^2 = 1$. Equations (8) and (9) show that

$$\mathbf{u} = (\xi, \eta, \zeta)\,(\rho_0 c)^{-1}(p - p_0), \tag{22}$$

again signifying longitudinal waves with velocity component $(\rho_0 c)^{-1}(p-p_0)$ in the direction of propagation.

The independence of the wave speed c on the direction (ξ, η, ζ) as well as on the shape h of the waveform, is a simplifying feature of 'the wave equation' that disappears in many other problems of waves in fluids (see below, beginning with chapters 3 and 4).

1.2 The speed of sound

Although the formula (14) for the speed of sound was known already to Newton, he failed to obtain good agreement between its indications and the results of observations of sound speed. Boyle's experiments on gases had shown that moderate pressure increases so decrease gas volume that pressure is closely proportional to density at a fixed temperature, suggesting that (14) can be written $c^2 = p_0/\rho_0$, which in atmospheric air at 20 °C gives $c = 290\,\mathrm{m\,s^{-1}}$, significantly lower than the observed value of $340\,\mathrm{m\,s^{-1}}$.

Not till more than a century later did Laplace explain this shortfall as due to the inappropriateness of using data obtained at a fixed value of the temperature. Wherever in a sound wave a fluid element is being compressed, neighbouring fluid is doing work upon it and this 'work of compression' adds internal energy to the element and so raises its temperature. Experiments such as those of Boyle on gases are performed in a container of large heat capacity with which the gas is allowed to exchange heat after compression, the volume change being measured only after reaching a steady state associated with return to the initial temperature.

By contrast, for the local compressions inside a sound wave there is no such restraint upon the temperature rise, whose value and whose effect on the speed of sound we now calculate for those gases which do to close approximation satisfy Boyle's law. These are the so-called 'perfect gases', including atmospheric air and any gas whose density is very small compared with that of the same substance in a condensed phase. Then the pressure takes to good approximation the value

$$p = RT\rho, \qquad (23)$$

where T is the absolute temperature in kelvins (that is, 273 plus the temperature in °C) and where the formula

$$R = \frac{8314\,\mathrm{m^2\,s^{-2}\,K^{-1}}}{\text{mean molecular weight}} \qquad (24)$$

gives RT in $\mathrm{m^2\,s^{-2}}$ in terms of the mean molecular weight of the gas or gas mixture.

Physically, a perfect gas is one in which at each instant only a very small proportion of the molecules are sufficiently close to others to be interacting with them. Its pressure is close to the value $RT\rho$, associated with momentum transfer by the random translational motions of molecules, because contributions proportional to higher powers of ρ from intermolecular forces are negligible. Its internal energy E per unit mass is to close approximation a function $E(T)$ of the temperature alone, proportional to the average energy (translational, rotational and vibrational) of an isolated molecule, because contributions from the potential energy associated with intermolecular forces are again negligible.

Whenever the volume of an element of gas is unchanging, any rise in temperature $\mathrm{d}T$ demands, per unit mass, a heat input equal to the required increase $E'(T)\,\mathrm{d}T$ in the internal energy. For this reason $E'(T)$ is written c_v, the *specific heat* (heat input per unit mass per unit increase in temperature) *at constant volume.*

There is an additional rise in internal energy (due to input of *work* rather than heat) when a fluid element is compressed, so that its volume, say V, changes by an amount $\mathrm{d}V$ which is negative. The total work done on it by the pressure p of adjacent elements is then $p(-\mathrm{d}V)$, since every adjacent small element does a quantity of work (that is, the *force* with which it acts *times the displacement* in the direction of that force) per unit reduction in volume equal to the force divided by the area of application; that is, to the pressure p. But the volume V per unit mass is ρ^{-1}, and so the energy change per unit mass due to work of compression is

$$\mathrm{d}E = p(-\mathrm{d}\rho^{-1}) = p\rho^{-2}\mathrm{d}\rho. \tag{25}$$

When this is the only source of internal energy change, as in sound waves when both *conduction* of heat and its generation by *dissipation* of mechanical energy can be neglected, the corresponding temperature rise $\mathrm{d}T$ is given by

$$p\rho^{-2}\mathrm{d}\rho = \mathrm{d}E = c_v\,\mathrm{d}T = (c_v/R)\,(\rho^{-1}\mathrm{d}p - p\rho^{-2}\mathrm{d}\rho), \tag{26}$$

where (23) has been used to relate temperature changes to changes in pressure and density.

The square of the velocity of sound, c^2, given by (14) as $\mathrm{d}p/\mathrm{d}\rho$, is now seen, for the pressure–density relations (26) characteristic of sound waves in perfect gases, to take the value

$$c^2 = \gamma p/\rho = \gamma RT, \tag{27}$$

where for perfect gases

$$\gamma = (R + c_v)/c_v. \tag{28}$$

Expression (27) is Newton's value p/ρ multiplied by γ, which for atmospheric air takes the value 1.40. We obtain the corresponding value of R from (24), using a mean molecular weight of 29.0, and infer at a temperature $T = 293$ K (corresponding to 20 °C) $c = 340\ \mathrm{m\,s^{-1}}$ to two significant figures, as observed.

For perfect gases in general, the quantity (28) can always be written

$$\gamma = c_p/c_v, \tag{29}$$

where c_p, the specific heat *at constant pressure*, is necessarily equal to $c_v + R$. That is because gas, on being heated at constant pressure (for example, under a given column of mercury), *expands*, doing work on adjacent fluid equal to $p\mathrm{d}\rho^{-1}$ per unit mass (namely, minus the work (25) done by adjacent fluid on the element), and this is $R\mathrm{d}T$ for changes at constant pressure by (23). Both this work $R\mathrm{d}T$ and the rise $c_v\,\mathrm{d}T$ in internal energy must be supplied by the total heat input $c_p\,\mathrm{d}T$.

Typical values of γ range from a maximum of $\frac{5}{3}$ for monatomic gases, whose internal energy is simply the translational energy of the molecules (contributing an amount $\frac{3}{2}R$ to c_v), through $\frac{7}{5}$ for diatomic gases (possessing an additional contribution of R to c_v from the energy of molecular rotation), to values as low as 1.2 or even 1.1 for polyatomic gases at high temperature (with a large further contribution to c_v from molecular vibrations). In all these cases the formula (27), stating that the square of the sound speed, c^2, exceeds its Newtonian value by the factor γ, is well borne out by observations.

For liquids, or for gases that are too dense to be regarded as 'perfect gases', the simple equation of state (23) and functional relationship between E and T have to be replaced by more complicated relationships

$$p = p(\rho, T), \quad E = E(\rho, T), \tag{30}$$

but c^2 is still given as the value of $\mathrm{d}p/\mathrm{d}\rho$ in changes satisfying (25). Rather surprisingly, it still turns out that

$$c^2 = \gamma c_{\mathrm{N}}^2, \tag{31}$$

where the 'Newtonian' value c_{N}^2 is $\partial p/\partial \rho$ keeping T constant, and where equation (29) still specifies γ as the ratio of the two specific heats.

This is because any small temperature change $\mathrm{d}T$ at constant pressure produces a density change

$$\mathrm{d}\rho = -\frac{\partial p/\partial T}{\partial p/\partial \rho}\,\mathrm{d}T, \tag{32}$$

and so the heat input $\mathrm{d}E - p\rho^{-2}\mathrm{d}\rho$ at constant pressure bears to its constant-volume value $(\partial E/\partial T)\,\mathrm{d}T$ the ratio

$$\gamma = \left[\frac{\partial E}{\partial T} + \left(\frac{\partial E}{\partial \rho} - p\rho^{-2}\right)\left(-\frac{\partial p/\partial T}{\partial p/\partial \rho}\right)\right] \Bigg/ \frac{\partial E}{\partial T}. \tag{33}$$

On the other hand, a density change satisfying (25) implies a temperature change

$$\mathrm{d}T = \frac{p\rho^{-2} - \partial E/\partial \rho}{\partial E/\partial T}\,\mathrm{d}\rho, \tag{34}$$

and so the alternative definition (31) of γ gives

$$\gamma = \left[\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial T}\left(\frac{p\rho^{-2} - \partial E/\partial \rho}{\partial E/\partial T}\right)\right] \Bigg/ \frac{\partial p}{\partial \rho}, \tag{35}$$

which evidently is identical with (33).

A more fundamental insight into the nature of the sound speed, however, is given by general thermodynamic theory through the concept of 'entropy'.

For a full account of entropy and its properties, see texts on thermodynamics and statistical physics; here, only those properties most needed for the study of waves in fluids are briefly summarised.

Entropy is a quantity that remains constant in any 'reversible' process, like the changes postulated above as occurring in sound waves. The internal energy E per unit mass changes exactly by the amount (25) in a reversible process; there is no extra change due to dissipation of kinetic energy into heat or due to transfer of heat from outside; also, while the process takes place, the fluid continues to satisfy the same relationships (30) as would characterise equilibrium conditions. Thus, a reversible process is one which so avoids *abrupt gradients* in both space and time that (i) viscous dissipation of energy into heat and conduction of heat produce negligible effects and (ii) the distribution of heat energy as between different modes of molecular motion remains always close to an equilibrium distribution. It is reversible because an equal and opposite change through the same set of equilibrium states restores the initial condition.

It follows that, when gradients in a sound wave are not too abrupt, the entropy per unit mass S remains constant. Accordingly, if a relationship

$$p = p(\rho, S) \tag{36}$$

for a particular fluid in equilibrium conditions can be derived, then

$$c^2 = \partial p/\partial \rho, \tag{37}$$

where this partial derivative now signifies a derivative keeping S constant.

The entropy, however, is not just any quantity constant in a reversible process (a process which in particular satisfies (25)); it and the absolute temperature T are so defined that any departure from (25) in a small change between equilibrium states satisfies

$$\mathrm{d}E - p\rho^{-2}\mathrm{d}\rho = T\mathrm{d}S; \tag{38}$$

in other words, the net additional heat absorbed is $T\mathrm{d}S$. When all quantities are written as functions of ρ and T as in (30), equation (38) implies that

$$\frac{\partial S}{\partial \rho} = \frac{1}{T}\left(\frac{\partial E}{\partial \rho} - p\rho^{-2}\right), \quad \frac{\partial S}{\partial T} = \frac{1}{T}\frac{\partial E}{\partial T}. \tag{39}$$

Hence, expressing that $\partial^2 S/\partial\rho\partial T = \partial^2 S/\partial T\partial\rho$, we obtain Maxwell's relationship

$$\partial E/\partial\rho = (p - T\partial p/\partial T)\rho^{-2}, \tag{40}$$

which can be used in (33) to give another representation of γ in terms of measurable quantities:

$$\gamma = 1 + (\alpha^2 T c_N^2/c_v), \tag{41}$$

where it is necessary to explain that α is the fluid's coefficient of expansion, $(-\mathrm{d}\rho/\rho\mathrm{d}T)$ for changes at constant pressure; this implies that

$$\mathrm{d}p = c_{\mathrm{N}}^2(\mathrm{d}\rho+\alpha\rho\mathrm{d}T) \tag{42}$$

for general changes and thus gives the values of $\partial p/\partial\rho$ and $\partial p/\partial T$ inserted in (33) and (40) to derive (41).

The quantity γ always exceeds 1 but does so for most liquids by a considerably smaller margin than for most gases. The margin is exceptionally small for cold water with its unusually low value of the product αT (a value which actually vanishes at $T = 277$ K). Values of αT more typical of liquids in general are exhibited by hot water; the product rises to 0.27 when $T = 371$ K (that is, 98 °C), giving $\gamma = 1.10$. The sound speed c takes values about 1400 m s^{-1} in water, and values of similar magnitude in most liquids.

The second law of thermodynamics states that the *total entropy* in any thermally isolated system can never decrease. Either the processes occurring are reversible and the entropy remains constant, or they deviate from reversibility and the entropy increases; as, for example, when a heat input $T\mathrm{d}S$ per unit mass results from viscous dissipation of kinetic energy, or when a certain quantity of heat is transferred from a part of the system with higher T to a part with lower T (so that the latter gains more entropy than the former loses). From the point of view of statistical physics, entropy is a measure of the randomness of the molecular organisation of the substance and this measure of randomness remains constant in reversible changes, while, however, its total value for an isolated system can only increase in irreversible changes as the system moves into regions of 'state space' with greater and greater probability....

In sound waves, any irreversible processes neglected in section 1.1, including viscosity and heat conduction, must in this way produce increases in the total entropy, corresponding to a heating of the fluid through which the sound wave passes and a corresponding gradual dissipation of the mechanical energy of the sound wave (a quantity whose meaning is made precise in section 1.3). A quantitative investigation of this dissipation process is given in section 1.13.

At this stage we may also discuss briefly any possible effect on sound propagation of another feature neglected in section 1.1; namely, such an external force field as gravity. Its presence means that the pressure p_0 and density ρ_0 in the undisturbed fluid are not uniform, but rather satisfy the hydrostatic relationship

$$\nabla p_0 = \rho_0\,\mathbf{g}, \tag{43}$$

where $\mathbf{g}$ is the vector acceleration due to gravity.