Lectures on Cyclotomic Hecke Algebras

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1 Introduction

The purpose of these lectures is to introduce the audience to the theory of cyclotomic Hecke algebras of type $G(m,1,n)$. These algebras were introduced by the author and Koike, Broué and Malle independently. As is well known, group rings of Weyl groups allow certain deformation. It is true for Coxeter groups, which are generalization of Weyl groups. These algebras are now known as (Iwahori) Hecke algebras.

Less studied is its generalization to complex reflection groups. As I will explain later, this generalization is not artificial. The deformation of the group ring of the complex reflection group of type $G(m,1,n)$ is particularly successful. The theory uses many aspects of very modern development of mathematics: Lusztig and Ginzburg's geometric treatment of affine Hecke algebras, Lusztig's theory of canonical bases, Kashiwara's theory of global and crystal bases, and the theory of Fock spaces which arises from the study of solvable lattice models in Kyoto school.

This language of Fock spaces is crucial in the theory of cyclotomic Hecke algebras. I would like to mention a little bit of history about Fock spaces in the context of representation theoretic study of solvable lattice models. For level one Fock spaces, it has origin in Hayashi's work. The Fock space we use is due to Misra and Miwa. For higher level Fock spaces, they appeared in work of Jimbo, Misra, Miwa and Okado, and Takemura and Uglov. We also note that Varagnolo and Vasserot's version of level one Fock spaces have straight generalization to higher levels and coincide with the Takemura and Uglov's one. The Fock spaces we use are different from them. But they are essential in the proofs.

Since the cyclotomic Hecke algebras contain the Hecke algebras of type A and type B as special cases, the theory of cyclotomic Hecke algebras is also useful to study the modular representation theory of finite classical groups of Lie type.

I shall explain theory of Dipper and James, and its relation to our theory. The relevant Hecke algebras are Hecke algebras of type A. In this case, we have an alternative approach depending on the Lusztig's conjecture on quantum groups, by virtue of Du's refinement of Jimbo's Schur-Weyl reciprocity. Even for this rather well studied case, our viewpoint gives a new insight. This

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viewpoint first appeared in work of Lascoux, Leclerc and Thibon. This Fock space description looks quite different from the Kazhdan-Lusztig combinatorics, since it hides affine Kazhdan-Lusztig polynomials behind the scene. Inspired by this description, Goodman and Wenzl have found a faster algorithm to compute these polynomials. Leclerc and Thibon are key players in the study of this type A case. I also would like to mention Schiffman and Vasserot's work here, since it makes the relation of canonical bases between modified quantum algebras and quantized Schur algebras very clear.

I will refer to work of Geck, Hiss, and Malle a little if time allows, since we can expect future development in this direction. It is relevant to Hecke algebras of type B. Finally, I will end the lectures with Broué's famous dream.

Detailed references can be found at the end of these lectures. The first three are for overview, and the rest are selected references for the lectures. [i-] implies a reference for the i th lecture.

2 Lecture One

2.1 Definitions

Let k be a field (or an integral domain in general). We define cyclotomic Hecke algebras of type $G(m,1,n)$ as follows.

Definition 2.1 *Let $v_1,\dots,v_m,q$ be elements in k, and assume that q is invertible. The Hecke algebra $\mathcal{H}_n(v_1,\dots,v_m;q)$ of type $G(m,1,n)$ is the k-algebra defined by the following relations for generators a_i $(1\le i\le n)$. We often write $\mathcal{H}_n$ instead of $\mathcal{H}_n(v_1,\dots,v_m;q)$. If we want to make the base ring explicit, we write $\mathcal{H}_n/k$.*

$$(a_1-v_1)\cdots(a_1-v_m)=0, \qquad (a_i-q)(a_i+1)=0 \quad (i\ge 2)$$

$$a_1a_2a_1a_2=a_2a_1a_2a_1, \qquad a_ia_j=a_ja_i \quad (j\ge i+2)$$

$$a_ia_{i-1}a_i=a_{i-1}a_ia_{i-1}\ (3\le i\le n)$$

The elements $L_i=q^{1-i}a_ia_{i-1}\cdots a_2a_1a_2\cdots a_i$ $(1\le i\le n)$ are called (Jucy-) Murphy elements or Hoefsmit elements.

[0]I would like to thank all the researchers involved in the development. Good interaction with German modular representation group (Geck, Hiss, Malle; Dipper), British combinatorial modular representation group (James, Mathas, Murphy), French combinatorics group (Lascoux, Leclerc, Thibon), modular representation group (Broué, Rouquier; Vigneras), geometric representation group (Varagnolo, Vasserot, Schiffman) and Kyoto solvable lattice model group (Okado, Takemura, Uglov) has nourished the rapid development. We still have some problems to solve, and welcome young people who look for problems.

I also thank Kashiwara, Lusztig, Ginzburg for their theories which we use.

Remark 2.2 *Let $\hat{H}_n$ be the (extended) affine Hecke algebra associated with the general linear group over a non-archimedian field. For each choice of positive root system, we have Bernstein presentation of this algebra. Let $P = \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_n$ be the weight lattice as usual. We adopt "geometric choice" for the positive root system. Namely $\{\alpha_i := \epsilon_{i+1} - \epsilon_i\}$ are simple roots. Let S be the associated set of Coxeter generators (simple reflections). Then $\hat{H}_n$ has description via generators X_ϵ ($\epsilon \in P$) and T_s ($s \in S$). We omit the description since it is well known. The following mapping gives rise a surjective algebra homomorphism from $\hat{H}_n$ to $\mathcal{H}_n$.*

$$X_{\epsilon_i} \mapsto L_i, \qquad T_{s_{\alpha_i}} \mapsto a_{i+1}$$

This fact is the reason why we can apply Lusztig's theory to the study of cyclotomic Hecke algebras. Since the module theory for $\mathcal{H}_n$ has been developed by different methods, it has also enriched the theory of affine Hecke algebras.

Remark 2.3 *Let ζ_m be a primitive m th root of unity. If we specialize $q = 1, v_i = \zeta_m^{i-1}$, we have the group ring of $G(m,1,n)$. $G(m,1,n)$ is the group of $n \times n$ permutation matrices whose non zero entries are allowed to be m th roots of unity. Under this specialization, L_i corresponds to the diagonal matrix whose i th diagonal entry is ζ_m and whose remaining diagonal entries are 1. We would like to stress two major differences between the group algebra and the deformed algebra $\mathcal{H}_n$.*

(1) $(L_i - v_1)\cdots(L_i - v_m)$ is not necessarily zero for $i > 1$.

(2) If we consider the subalgebra generated by Murphy elements, its dimension is not m^n in general. Further, the dimension depends on parameters $v_1, \ldots, v_m, q$.

Nevertheless, we have the following Lemma. a_w is defined by $a_{i_1} \cdots a_{i_l}$ for a reduced word $s_{i_1} \cdots s_{i_l}$ of w. It is known that a_w does not depend on the choice of the reduced word.

Lemma 2.4 *$\{L_1^{e_1} \cdots L_n^{e_n} a_w | 0 \le e_i < m, w \in \mathfrak{S}_n\}$ form a basis of $\mathcal{H}_n$.*

(How to prove) We consider $\mathcal{H}_n$ over an integral domain R, and show that $\sum RL_1^{e_1} \cdots L_n^{e_n} a_w$ is a two sided ideal. Then we have that these elements generate $\mathcal{H}_n$ as an R-module. To show that they are linearly independent, it is enough to take $R = \mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}, \mathbf{v}_1, \ldots, \mathbf{v}_m]$. In this generic parameter case, we embed the algebra into $\mathcal{H}_n/\mathbb{Q}(\mathbf{q}, \mathbf{v}_1, \ldots, \mathbf{v}_m)$. Then we can construct enough simple modules to evaluate the dimension. ■

An important property of $\mathcal{H}_n$ is the following.

Theorem 2.5 (Malle-Mathas) *Assume that v_i are all invertible. Then $\mathcal{H}_n$ is a symmetric algebra.*

(How to prove) Since $\mathcal{H}_n$ is deformation of the group algebra of $G(m,1,n)$, we can define a length function $l(w)$ and a_w for a reduced word of w. Unlike the Coxeter group case, a_w does depend on the choice of the reduced word. Nevertheless, the trace function

$$tr(a_w) = \begin{cases} 0 & (w \neq 1) \\ 1 & (w = 1) \end{cases}$$

is well defined. $(u,v) := tr(uv)$ $(u,v \in \mathcal{H}_n)$ gives the bilinear form with the desired properties. ■

Remark 2.6 *We have defined deformation algebras for (not all but most of) other types of irreducible complex reflection groups by generators and relations. ($G(m,p,n)$: the author, other exceptional groups: Broué and Malle.)*

The most natural definition of cyclotomic Hecke algebras is given by Broué, Malle and Rouquir. It coincides with the previous definition in most cases.

Let $\mathcal{A}$ be the hyperplane arrangement defined by complex reflections of W. For each $\mathcal{C} \in \mathcal{A}/W$, we can associate the order $e_{\mathcal{C}}$ of the cyclic group which fix a hyperplane in $\mathcal{C}$. Primitive idempotents of this cyclic group are denoted by $\epsilon_j(H)$ $(0 \leq j < e_{\mathcal{C}})$. We set $\mathcal{M} = \mathbb{C}^n \setminus \cup_{H \in \mathcal{A}} H$.

Definition 2.7 *For each hyperplane H, let α_H be the linear form whose kernel is H. It is defined up to scalar multiple. We fix a set of complex numbers $t_{\mathcal{C},j}$. Then the following partial differential equation for $\mathbb{C}W$-valued functions F on $\mathcal{M}$ is called the (generalized) KZ equation.*

$$\frac{\partial F}{\partial x_i} = \frac{1}{2\pi\sqrt{-1}} \sum_{\mathcal{C} \in \mathcal{A}/W} \sum_{j=0}^{e_{\mathcal{C}}-1} \sum_{H \in \mathcal{C}} \frac{\partial(\log \alpha_H)}{\partial x_i} t_{\mathcal{C},j} \epsilon_j(H) F$$

Theorem 2.8 (Broué-Malle-Rouquier) *Assume that parameters are sufficiently generic. Let B be the braid group attached to $\mathcal{A}$. Then the monodromy representation of B with respect to the above KZ equation factors through a deformation ring of $\mathbb{C}W$. If $W = G(m,1,n)$ for example, it coincides with the cyclotomic Hecke algebra with specialized parameters.*

2.2 Representations

If all modules are projective modules, we say that $\mathcal{H}_n$ is a semi-simple algebra, and call these representations **ordinary representations**. We have

Proposition 2.9 (Ariki(-Koike)) *$\mathcal{H}_n$ is semi-simple if and only if $q^i v_j - v_k$ $(|i| < n, j \neq k)$ and $1 + q + \cdots + q^i$ $(1 \leq i < n)$ are all non zero. In this case, simple modules are parametrized by m-tuples of Young diagrams of* *total size n. For each* $\lambda = (\lambda^{(m)}, \ldots, \lambda^{(1)})$*, the corresponding simple module can be realized on the space whose basis elements are indexed by standard tableaux of shape* λ*. The basis elements are simultaneous eigenvectors of Murphy elements, and we have explicit matrix representation for generators* a_i $(1 \le i \le n)$*.*

These represenations are called **semi-normal form representations**. Hence we have complete understanding of ordinary representations. If $\mathcal{H}_n$ is not semi-simple, representations are called **modular representations**. A basic tool to get information for modular representations from ordinary ones is "reduction" procedure.

Definition 2.10 *Let* (K, R, k) *be a modular system. Namely, R is a discrete valuation ring, K is the field of fractions, and k is the residue field. For an* $\mathcal{H}_n/K$*-module V, we take an* $\mathcal{H}_n/R$*-lattice* V_R *and set* $\overline{V} = V_R \otimes k$*. It is known that* $\overline{V}$ *does depend on the choice of* V_R*, but the composition factors do not depend on the choice of* V_R*. The map between Grothendieck groups of finite dimentional modules given by*

$$dec_{K,k} : K_0(mod\text{–}\mathcal{H}_n/K) \longrightarrow K_0(mod\text{–}\mathcal{H}_n/k)$$

which sends $[V]$ *to* $[\overline{V}]$ *is called a* **decomposition map***. Since Grothendieck groups have natural basis given by simple modules, we have the matrix representation of the decomposition map with respect to these bases. It is called the* **decomposition matrix***. The entries are called* **decomposition numbers***.*

In the second lecture, we also consider the decomposition map between Grothendieck groups of $KGL(n, q)$*–mod* and $kGL(n, q)$*–mod*.

Remark 2.11 *Decomposition maps are not necessarily surjective even after coefficients are extended to complex numbers. If we take* $m = 1, 2$ *and* $q \in k$ *to be zero, we have counter examples. These are called zero Hecke algebras, and studied by Carter.* **Note that we exclude the case** $q = 0$ **in the definition***. In the case of group algebras, the theory of Brauer characters ensures that decomposition maps are surjective.*

In the case of cyclotomic Hecke algebras, we have the following result.

Theorem 2.12 (Graham-Lehrer) $\mathcal{H}_n$ *is a cellular algebra. In particular, the decomposition maps are surjective.*

The notion of cellularity is introduced by Graham and Lehrer. It has some resemblance to the definition of quasi hereditary algebras. This is further pursued by König and Changchang Xi.

In this lecture, we follow Dipper, James and Mathas' construction of Specht modules. We first fix notation.

Let $\lambda = (\lambda^{(m)}, \dots, \lambda^{(1)})$, $\mu = (\mu^{(m)}, \dots, \mu^{(1)})$ be two m-tuples of Young diagrams. We say λ dominates μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{j>k} |\lambda^{(j)}| + \sum_{j=1}^{l} \lambda_j^{(k)} \geq \sum_{j>k} |\mu^{(j)}| + \sum_{j=1}^{l} \mu_j^{(k)}$$

for all k, l. This partial order is called **dominance order**.

For each $\lambda = \big(\lambda^{(m)}, \dots, \lambda^{(1)}\big)$, we set $a_k = n - |\lambda^{(1)}| - \cdots - |\lambda^{(k)}|$. We have $n \geq a_1 \geq \cdots \geq a_l > 0$ and $a_k = 0$ for $k > l$ for some l. we denote l by $l(a)$. For $a = (a_k)$, we denote by $\mathfrak{S}_a$ the set of permutations which preserve $\{1, \dots, a_l\}, \dots, \{a_k + 1, \dots, a_{k-1}\}, \dots \{a_1 + 1, \dots, n\}$. We also set

$$\begin{aligned} u_a = \ & (L_1 - v_1) \cdots (L_{a_1} - v_1) \times (L_1 - v_2) \cdots (L_{a_2} - v_2) \times \cdots \\ & \cdots \times (L_1 - v_{l(a)}) \cdots (L_{l(a)} - v_{l(a)}) \end{aligned}$$

Let t^λ be the canonical tableau. It is the standard tableau on which $1, \dots, n$ are filled in by the following rule;
$1, \dots, \lambda_1^{(m)}$ are written in the first row of $\lambda^{(m)}$; $\lambda_1^{(m)} + 1, \dots, \lambda_1^{(m)} + \lambda_2^{(m)}$ are written in the second row of $\lambda^{(m)}$; $\dots$; $|\lambda^{(m)}|+1, \dots, |\lambda^{(m)}|+\lambda_1^{(m-1)}$ are written in the first row of $\lambda^{(m-1)}$; and so on.

The row stabilizer of t^λ is denoted by $\mathfrak{S}_\lambda$. We set

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} a_w, \quad m_\lambda = x_\lambda u_a = u_a x_\lambda.$$

Let t be a standard tableau of shape λ. If the location of $i_k \in \{1, \dots, n\}$ in t is the same as the location of k in t^λ, We define $d(t) \in \mathfrak{S}_n$ by $k \mapsto i_k$ $(1 \leq k \leq n)$.

Definition 2.13 *Let $* : \mathcal{H}_n \to \mathcal{H}_n$ be the anti-involution induced by $a_i^* = a_i$. For each pair (s,t) of standard tableaux of shape λ, we set $m_{st} = a_{d(s)}^* m_\lambda a_{d(t)}$.*

Remark 2.14 *$\{m_{st}\}$ form a cellular basis of $\mathcal{H}_n$.*

Proposition 2.15 (Dipper-James-Mathas) *Let (K, R, k) be a modular system. We set $\mathcal{I}_\lambda = \sum R m_{st}$ where sum is over pairs of standard tableaux of shape strictly greater than λ (with respect to the dominance order). Then $\mathcal{I}_\lambda$ is a two sided ideal of $\mathcal{H}_n/R$.*

(How to prove) It is enough to consider straightening laws for elements $a_i m_{st}$ and $m_{st} a_i$. We can then show that m_{uv} appearing in the expression have greater shapes with respect to the dominance order. ■

Definition 2.16 *Set $z_\lambda = m_\lambda \bmod \mathcal{I}_\lambda$. Then the submodule $S^\lambda = z_\lambda \mathcal{H}_n$ of $\mathcal{H}_n/\mathcal{I}_\lambda$ is called a* **Specht module**.

Theorem 2.17 (Dipper-James-Mathas)
$\{z_\lambda a_{d(t)} |\ t : \textit{standard of shape } \lambda\}$ *form a basis of* S^λ.

(How to prove) We can show by induction on the dominance order that these generate S^λ. Hence the collection of all these generate $\mathcal{H}_n$. Thus counting argument completes the proof. ■

Definition 2.18 S^λ *is equipped with a bilinear form defined by*

$$\langle z_\lambda a_{d(t)}, z_\lambda a_{d(s)} \rangle m_\lambda = m_\lambda a_{d(s)} a^*_{d(t)} m_\lambda \bmod \mathcal{I}_\lambda$$

Theorem 2.19 (General theory of Specht modules)
(1) $D^\lambda = S^\lambda / \mathrm{rad}\langle\ ,\ \rangle$ *is absolutely irreducible or zero module.* $\{D^\lambda \neq 0\}$ *form a complete set of simple* $\mathcal{H}_n$*- modules.*
(2) Assume $D^\mu \neq 0$ *and* $[S^\lambda : D^\mu] \neq 0$*. Then we have* $\mu \trianglelefteq \lambda$.

Remark 2.20 *In the third lecture, we give a criterion for non vanishing of* D^λ.

Theorem 2.21 (Dipper-Mathas) *Let* $\{v_1, \ldots, v_m\} = \sqcup_{i=1}^a S_i$ *be the decomposition such that* v_j, v_k *are in a same* S_i *if and only if* $v_j = v_k q^b$ *for some* $b \in \mathbb{Z}$*. Then we have*

$$\textit{mod}\text{–}\mathcal{H}_n \simeq \bigoplus_{n_1,\ldots,n_a} \textit{mod}\text{–}\mathcal{H}_{n_1} \boxtimes \cdots \boxtimes \textit{mod}\text{–}\mathcal{H}_{n_a}$$

where $\mathcal{H}_n = \mathcal{H}_n(v_1, \ldots, v_m; q)$, $\mathcal{H}_{n_i} = \mathcal{H}_{n_i}(S_i; q)$, *and the sum runs through* $n_1 + \cdots + n_a = n$.

Hence, it is enough to consider the case that v_i are powers of q.

Remark 2.22 *For the classification of simple modules, we can use arguments of Rogawski and Vigneras for the reduction to the case that* v_i *are powers of* q*. Hence we do not need the above theorem for this purpose.*

2.3 First application

Let $k_q^\times = k^\times / \langle q \rangle$. We assume that $q \neq 1$, and denote the multiplicative order of q by r. A **segment** is a finite sequence of consecutive residue numbers which take values in $\mathbb{Z}/r\mathbb{Z}$. A **multisegment** is a collection segments. Assume that a multisegment is given. Take a segment in the multisegment. By adding i ($i \in \mathbb{Z}/r\mathbb{Z}$) to the entries of the segment simultaneously, we have a segment of shifted entries. If all of these r segments appear in the given multisegment, we say that the given multisegment is **periodic**. If it never happens for all segements in the multisegment, we say that the given multisegment is **aperiodic**. We denote by $\mathcal{M}_r^{ap}$ the set of aperiodic multisegments.

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Theorem 2.23 (Ariki-Mathas) *Simple modules over $\hat{H}_n/k$ are parametrized by*

$$\mathcal{M}_r^{ap}(k) = \{\lambda : k_q^\times \to \mathcal{M}_r^{ap} | \sum_{x \in k_q^\times} |\lambda(x)| = n\}$$

(How to prove) We consider a setting for reduction procedure, and show that a lower bound and an upper bound for the number of simple modules coincide. To achieve the lower bound, we use the integral module structure of the direct sum of Grothendieck groups of $proj\text{-}\mathcal{H}_n$ with respect to a Kac-Moody algebra action, which will be explained in the second lecture. The upper bound is achieved by cellularity. ■

Remark 2.24 *The lower bound can be achieved by a different method. This is due to Vigneras.*

Let F be a nonarchimedian local field and assume that the residue field has characteristic different from the characteristic of k. We assume that k is algebraically closed. We consider admissible k-representations of $GL(n, F)$. We take modular system (K, R, k) and consider reduction procedure.

Theorem 2.25 (Vigneras) *All cuspidal representations are obtained by reduction procedure. The admissible dual of k-representations is obtained from the classification of simple $\hat{H}_n/k$-modules.*

Hence we have contribution to the last step of the classification.

Remark 2.26 *Her method is induction from open compact groups and theory of minimal K-types. In the characteristic zero case, it is done by Bushnell and Kutzko. Considering $M := ind_{G,K}(\sigma)$ where (K, σ) is irreducible cuspidal distinguished K-type, she shows that $\mathrm{End}_{kG}(M)$ is isomorphic to product of affine Hecke algebras, and M satisfies the following hypothesis.*

"There exists a finitely generated projective module P and a surjective homomorphism $\beta : P \to M$ such that $\mathrm{Ker}(\beta)$ is $\mathrm{End}_{kG}(P)$-stable."

Then the classification of simple kG-modules reduces to that of simple $\mathrm{End}_{kG}(M)$-modules. This simple fact is known as Dipper's lemma.

3 Lecture Two

3.1 Geometric theory

Let $\mathcal{N}$ be the set of $n \times n$ nilpotent matrices, $\mathcal{F}$ be the set of n-step complete flags in $\mathbb{C}^n$. We define the **Steinberg variety** as follows.

$$Z = \{(N, F_1, F_2) \in \mathcal{N} \times \mathcal{F} \times \mathcal{F} | F_1, F_2 \text{ are } N\text{-stable}\}$$

$G := GL(n,\mathbb{C}) \times \mathbb{C}^\times$ naturally acts on Z via

$$(g,q)(N,F_1,F_2) = (q^{-1}Ad(g)N, gF_1, gF_2).$$

Let $K^G(Z)$ be the Grothendieck group of G-equivariant coherent sheaves on Z. It is an $\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]$- algebra via convolution product.

Theorem 3.1 (Ginzburg)
(1) We have an algebra isomorphism $K^G(Z) \simeq \hat{H}_n$.
(2) Let us consider a central character of the center $\mathbb{Z}[X^{\pm}_{\epsilon_1},\dots,X^{\pm}_{\epsilon_n}]^{\mathfrak{S}_n}[\mathbf{q}^{\pm}]$ *induced by* $\hat{s} : X_{\epsilon_i} \mapsto \lambda_i$. *By specializing the center via this linear character, we obtain a specialized affine Hecke algebra. Let* s *be* $\mathrm{diag}(\lambda_1,\dots,\lambda_n)$. *Then* $H_*(Z^{(s,q)},\mathbb{C})$ *equipped with convolution product is isomorphic to the specialized affine Hecke algebra. Here the homology groups are Borel-Moore homology groups, and* $Z^{(s,q)}$ *are fixed points of* $(s,q) \in G$.

Remark 3.2 *All simple modules are obtained as simple modules of various specialized affine Hecke algebras.*

Theorem 3.3 (Sheaf theoretic interpretation)
Let $\tilde{\mathcal{N}}$ *be* $\{(N,F) \in \mathcal{N} \times \mathcal{F} | F \text{ is } N\text{-stable}\}$, $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ *be the first projection. Then*

(1) $H_*(Z^{(s,q)},\mathbb{C}) \simeq Ext^*(\mu_*\mathbb{C}_{\tilde{\mathcal{N}}^{(s,q)}}, \mu_*\mathbb{C}_{\tilde{\mathcal{N}}^{(s,q)}})$.
(2) Let $\mu_*\mathbb{C}_{\tilde{\mathcal{N}}^{(s,q)}} = \oplus_{\mathcal{O}} \oplus_{k\in\mathbb{Z}} L_{\mathcal{O}}(k) \otimes IC(\mathcal{O},\mathbb{C})[k]$. *Then* $L_{\mathcal{O}} := \oplus_{k\in\mathbb{Z}} L_{\mathcal{O}}(k)$ *is a simple* $H_*(Z^{(s,q)},\mathbb{C})$*-module or zero module. Further, non-zero ones form a complete set of simple* $H_*(Z^{(s,q)},\mathbb{C})$*-modules. If* q *is not a root of unity, all* $L_{\mathcal{O}}$ *are non-zero. If* q *is a primitive* r *th root of unity,* $L_{\mathcal{O}} \neq 0$ *if and only if* $\mathcal{O}$ *corresponds to a (tuple of) aperiodic multisegments taking residues in* $\mathbb{Z}/r\mathbb{Z}$.

In the above theorem, the orbits run through orbits consisting of isomorphic representations of a quiver, which is disjoint union of infinite line quivers or cyclic quivers of length r. The reason is that $\mathcal{N}^{(s,q)}$ is the set of nilpotent matrices N satisfying $sNs^{-1} = qN$, which can be identified with representations of a quiver via considering eigenspaces of s as vector spaces on nodes and N as linear maps on arrows. This is the key fact which relates the affine quantum algebra of type A_∞, $A^{(1)}_{r-1}$ and representations of cyclotomic Hecke algebras.

Definition 3.4 *Let* $\mathcal{C}_n$ *be the full subcategory of mod–*$\hat{H}_n$ *whose objects are modules which have central character* $\hat{s}$ *with all eigenvalues of* s *being powers of* q. *Let* z *be an indeterminate and set* $c_n(z) = (z - X_{\epsilon_1})\cdots(z - X_{\epsilon_n}$. *We denote by* $P_{c_n(z),(z-q^{i_1})\cdots(z-q^{i_n})}(-)$ *the exact functor taking generalized eigenspaces of eigenvalue* $(z-q^{i_1})\cdots(z-q^{i_n})$ *with respect to* $c_n(z)$. *We then set*

$$i - Res(M) = \bigoplus_{f(z)\in k[z]} P_{c_{n-1}(z),f(z)/(z-q^i)}\left(Res^{\hat{H}_n}_{\hat{H}_{n-1}}\left(P_{c_n(z),f(z)}(M)\right)\right)$$

This is an exact functor from $\mathcal{C}_n$ *to* $\mathcal{C}_{n-1}$. *We set* $U_n = \mathrm{Hom}_{\mathbb{C}}(K_0(\mathcal{C}_n), \mathbb{C})$, $f_i = (i - Res)^T : U_{n-1} \to U_n$.

I shall give some historical comments here. The motivation to introduce these definitions was Lascoux-Leclerc-Thibon's observation that Kashiwara's global basis on level one modules computes the decomposition numbers of Hecke algebras of type A over the field of complex numbers. The above notions for affine Hecke algebras and cyclotomic Hecke algebras were first introduced by the author in his interpretation of Fock spaces and action of Chevalley generators in LLT observation into (graded dual of) Grothendieck groups of these Hecke algebras and i-restriction and i-induction operations. This is the starting point of a new point of view on the representation theory of affine Hecke algebras and cyclotomic Hecke algebras. As I will explain below, it allows us to give a new application of Lusztig's canonical basis. It triggered intensive studies of canonical bases on Fock spaces. These are carried out mostly in Paris and Kyoto. On the other hand, the research on cyclotomic Hecke algebras are mostly lead by Dipper, James, Mathas, Malle and the author. In the third lecture, these two will be combined to prove theorems on Specht module theory of cyclotomic Hecke algebras.

We now state a key proposition necessary for the proof of the next theorem. In the top row of the diagram, we allow certain infinite sum in $U(\mathfrak{g}(A_\infty))$ in accordance with infinite sum in U_n. Note that we do not have infinite sum in the bottom row.

Proposition 3.5 (Ariki) *There exists a commutative diagram*

$$\begin{array}{ccc} U^-(\mathfrak{g}(A_\infty)) & \simeq & \bigoplus_{n\ge 0} U_n/\mathbf{q} \\ \uparrow & & \uparrow \\ U^-(\mathfrak{g}(A^{(1)}_{r-1})) & \simeq & \bigoplus_{n\ge 0} U_n/q{=}\sqrt[r]{1} \end{array}$$

such that the left vertical arrow is inclusion, the right vertical arrow is induced by specialization $\mathbf{q} \to q$, *and the bottom horizontal arrow is an* $U^-(\mathfrak{g}(A^{(1)}_{r-1}))$*-module isomorphism. Under this isomorphism, canonical basis elements of* $U^-(\mathfrak{g}(A^{(1)}_{r-1}))$ *map to dual basis elements of* $\{[$*simple module*$]\}$.

(How to prove) We firstly construct the upper horizontal arrow by using PBW-type basis and dual basis of {[standard module]} of affine Hecke algebras. Here we use Kazhdan-Lusztig induction theorem. We also use restriction rule for Specht modules. We then appeal to folding argument. On the left hand side, we consider this folding in geometric terms. Since only short explanation was supplied in my original paper, I also refer to Varagnolo-Vasserot's argument for this part. Note that the Hall algebra of the cyclic quiver is realized as the vector space whose basis is given by infinite sums of dual basis elements of {[standard module]}. We then use

[standard module:simple module]=[canonical basis:PBW-type basis]