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Aspects of Sobolev-Type Inequalities

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Contents

Preface

These notes originated from a graduate course given at Cornell University during the fall of 1998. One of the aims of the course was to present Sobolev inequalities and some of their applications in the context of analysis on manifolds —including Harnack inequalities and heat kernel estimates— to an audience not necessarily very familiar with analysis in general and Sobolev inequalities in particular. The first part (Chapters 1–2) introduces the reader to Sobolev inequalities in $\mathbb{R}^n$. An important application, Moser's proof of the elliptic Harnack inequality for uniformly elliptic divergence form second order differential operators, is treated in detail. In the second part (Chapters 3–4), Sobolev inequalities on complete non-compact Riemannian manifolds are discussed: What is their meaning and when do they hold true? How does one prove them? This discussion is illustrated by the treatment of some explicit examples. In the third and last part, Chapter 5, families of local Sobolev and Poincaré inequalities are introduced. These turn out to be crucial for taking full advantage of Sobolev inequality techniques on Riemannian manifolds. For instance, complete Riemannian manifolds satisfying a scale-invariant parabolic Harnack inequality are characterized in terms of Poincaré inequalities and volume growth. These notes give the first detailed exposition of this fundamental result.

We warn the reader that no effort has been made to include a comprehensive bibliography. Many important papers related to the topics presented in these notes are not mentioned. Actually, the literature on Sobolev inequalities is so vast that it would certainly be difficult to list it all. A few of the classical books on the subject have been listed here.

Concerning Riemannian geometry, the books [5, 29] and [12, 13] are very useful references and contain some material related to the present text. There is some overlapping between these notes and the monographs [39, 40], but it may be less than one would think in view of the titles. In particular, the applications presented here and in [39, 40] are different.

Some of the techniques from functional analysis used here are developed in greater generality in [21, 72, 87]. Of these three books, the closest in spirit to these notes might be [21], although there is very little direct overlapping and the two complement each other. Grigor'yan's survey article [34] is a wonderful source of information for many related topics not treated in this monograph.

It is a pleasure to acknowledge the influence, direct or otherwise, that many colleagues and friends had on the writing of this text. Thanks to A. Ancona, D. Bakry, A. Bendikov, T. Coulhon, P. Diaconis, A. Grigor'yan, L. Gross, W. Hebisch, A. Hulanicki, M. Ledoux, N. Lohoué, M. Solomyak, D. Stroock and N. Varopoulos. Thanks to the students and colleagues at Cornell who attended the class on which these notes are based. They helped me to try to stay honest. Finally, I would like to thank the various institutions whose support over the years has made the writing of this book possible. They are, in no particular order, Le Centre National de la Recherche Scientifique, l'Université Paul Sabatier in Toulouse, France, the National Science Foundation (grant DMS-9802855), and Cornell University.