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Introduction

This introduction describes some of the main ideas, problems and techniques presented in this monograph.

Chapter 1 gives a brief but more or less self-contained account of Sobolev inequalities in $\mathbb{R}^n$. The Sobolev inequality in $\mathbb{R}^n$ asserts that

$$\left(\int_{\mathbb{R}^n} |f(x)|^{np/(n-p)}dx\right)^{np/(n-p)} \leq C(n,p)\left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx\right)^{1/p},$$

that is,

$$\|f\|_q \leq C(n,p)\|\nabla f\|_p, \quad q = np/(n-p),$$

for all smooth functions f with compact support and each $1 \leq p < n$. When $p > n$, the Hölder continuity estimate

$$\sup_{x,y\in\mathbb{R}^n}\left\{\frac{|f(x)-f(y)|}{|x-y|^{1-n/p}}\right\} \leq C\|\nabla f\|_p$$

holds instead. We discuss a number of different proofs of Sobolev inequalities in $\mathbb{R}^n$. Each yields a different and useful point of view on the meaning of Sobolev inequalities. Of course, this material is covered in greater detail in a number of books and monographs including [1, 30, 60, 79]. The important topic of Sobolev inequalities in subdomains of $\mathbb{R}^n$ (see, e.g., [61]) is not treated here.

The theory of partial differential equations provides a host of important applications of Sobolev inequalities. Consider for instance the equation

$$\sum_{i,j=1}^{n} \partial_i a_{i,j}(x)\partial_j u(x) = 0$$

where the coefficients $a_{i,j}$ are real measurable functions such that

$$\|a_{i,j}\|_\infty \leq C_1$$

and

$$\forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n, \quad \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq c_1 \sum_1^n \xi_i^2.$$

That is, consider a divergence form, uniformly elliptic equation in $\mathbb{R}^n$. Moser's elliptic Harnack inequality [30, 63] states that any positive weak solution u of this equation in an Euclidean ball B satisfies

$$\sup_{\frac{1}{2}B}\{u\} \leq C \inf_{\frac{1}{2}B}\{u\}$$

where C depends neither on u nor on B but only on the constants C_1, c_1 above and the dimension n. Moser's proof, presented in Chapter 2, is a striking application of Sobolev inequalities. It also serves as an introduction to our later treatment of parabolic Harnack inequalities on manifolds.

In Chapter 3, Sobolev inequalities are discussed in the context of Riemannian manifolds. A number of related functional inequalities are introduced and relations between these inequalities are established. One of the most basic facts is that any Sobolev inequality implies a lower bound on the volume growth of the geodesic balls. In particular, the inequality

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_q \leq C\|\nabla f\|_p$$

for some fixed $q > p \geq 1$, implies that the volume of any ball of radius r must be bounded below by a constant times r^ν with ν related to p, q by $1/\nu = 1/p - 1/q$.

A more technical but very important fact is the equivalence between strong forms and weak forms of Sobolev inequalities. An example of this phenomenon is that it is enough to have the weak Sobolev inequality

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \sup_{s>0}\left\{s\mu\left(\{x : |f(x)| > s\}\right)^{1/q}\right\} \leq C\|\nabla f\|_p$$

with $1 \leq p < q$ to conclude that the strong inequality $\|f\|_q \leq C\|\nabla f\|_p$ holds (with different constants C). Another example is the equivalence between the Nash inequality

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_2^{(1+2/\nu)} \leq C\|\nabla f\|_2 \|f\|_1^{2/\nu}$$

and the Sobolev inequality

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_{2\nu/(\nu-2)} \leq C\|\nabla f\|_2$$

when $\nu > 2$ (again with different C's). The Nash inequality is (a priori) weaker in the sense that it is easily deduced from the Sobolev inequality above and Hölder's inequality. Chapter 3 gives a rather complete treatment of this phenomenon using elementary and unified arguments taken from [6]. Related results and interesting developments concerning Sobolev spaces on metric spaces can be found in [38].

The equivalence between weak and strong forms of Sobolev-type inequalities turns out to be extremely useful when it comes to *prove* that a certain manifold satisfies a Sobolev inequality. This is illustrated in the last section of Chapter 3 where some fundamental examples are treated. A basic tool used here is the notion of pseudo-Poincaré inequality. Given a smooth function f, let $f_r(x)$ denote the mean of f over the ball of center x and radius r. One says that M satisfies an L^p-pseudo-Poincaré inequality if, for all $f \in \mathcal{C}_0^\infty(M)$ and all $r > 0$,

$$\|f - f_r\|_p \leq C\, r\, \|\nabla f\|_p.$$

For manifolds satisfying a pseudo-Poincaré inequality, Sobolev inequalities can be deduced from a simple lower bound on the volume growth. This is more precisely stated in the following theorem.

Theorem *Let M be a complete Riemannian manifold. Fix p, ν with $1 \leq p < \nu$ and assume that M satisfies an L^p-pseudo-Poincaré inequality. Then the Sobolev inequality*

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_{\nu p/(\nu - p)} \leq C\|\nabla f\|_p$$

holds true if and only if any ball B of radius $r > 0$ has volume bounded below by $\mu(B) \geq cr^\nu$.

The idea behind this theorem first appeared rather implicitly in [72] in the setting of Lie groups. It was later developed in [6, 19, 74] and other works. To illustrate this result, we treat in detail the case of unimodular Lie groups equipped with a left-invariant Riemannian metric as well as manifolds with non-negative Ricci curvature and maximal volume growth. The L^p-pseudo-Poincaré inequality should be compared with the more classical L^p-Poincaré inequality

$$\forall f \in \mathcal{C}^\infty(B), \quad \left(\int_B |f(y) - f_B|^p dy\right)^{1/p} \leq Cr \left(\int_B |\nabla f(y)|^p dy\right)^{1/p}$$

where $B = B(x, r)$ denotes a geodesic ball of radius r and $f_B = f_r(x)$ is the mean of f over B. This last inequality may or may not hold on M, uniformly over all balls $B = B(x, r)$, $x \in M$, $r > 0$. The pseudo-Poincaré inequality may hold for all $r > 0$ in cases where the Poincaré inequality does not (for instance on unimodular Lie groups having exponential volume growth).

Chapter 4 develops two different but related applications of Sobolev-type inequalities. These two applications have been chosen for their importance and their simplicity.

First, we show that Nash inequality is equivalent to a uniform heat kernel upper bound of the form

$$\sup_{x,y \in M} h(t, x, y) \leq Ct^{-\nu/2}$$

where $h(t,x,y)$ denotes the fundamental solution of the heat equation

$$(\partial_t + \Delta)u = 0$$

on $(0,\infty)\times M$, with $\Delta = -\mathrm{div}\circ\nabla$. In particular, under a Nash inequality, the heat diffusion semigroup $(H_t)_{t>0}$ is ultracontractive (i.e., sends L^1 to L^∞). This has been developed in the last fifteen years into a powerful machinery which produces Gaussian heat kernel upper bounds. Although this circle of ideas has its roots in Nash's 1958 paper [67], it was only after 1980 that the full strength and the scope of this technique was identified. The books [21, 72, 87] contain different accounts of this topic, various applications and further developments. Here, under the basic hypothesis that

$$\forall\, t>0, \qquad \sup_{x,y\in M} h(t,x,y) \le Ct^{-\nu/2},$$

we prove that the heat kernel satisfies the Gaussian upper bound

$$h(t,x,y) \le C_1 t^{-\nu/2}(1+d^2/t)^{\nu/2}e^{-d^2/4t}$$

where $d=d(x,y)$ is the Riemannian distance between x and y. Our proof is somewhat different from those found in the literature. It is adapted from [41] and uses complex interpolation as a main technical tool (and, ironically, no Sobolev-type inequality).

The second topic treated in Chapter 4 is a spectral inequality known as the Rozenblum–Lieb–Cwikel estimate. This inequality was first proved in $\mathbb{R}^n$ by Rozenblum in 1972. It asserts that the number of negative eigenvalues of the Schrödinger operator $\Delta - V$ is bounded above by $C(\nu)\|V_+\|_{\nu/2}^{\nu/2}$ as soon as the manifold M satisfies the Sobolev inequality

$$\|f\|_{2\nu/(\nu-2)} \le C\|\nabla f\|_2.$$

The proof presented here is due to P. Li and S-T. Yau, [55]. A central part of this proof is very close in spirit to Nash's ideas concerning ultracontractivity. It illustrates well what can be done by a skillful use of Sobolev inequality and basic functional analysis.

Despite important examples such as $\mathbb{R}^n$ and hyperbolic spaces, many Riemannian manifolds fail to satisfy a global Sobolev inequality of the form

$$\forall\, f\in \mathcal{C}_0^\infty(M), \quad \|f\|_{2\nu/(\nu-2)} \le C\|\nabla f\|_2$$

for some $\nu>2$. For one thing, such an inequality implies that the volume of any ball of radius r is at least cr^ν for all $r>0$, ruling out many simple interesting manifolds such as $\mathbb{S}^m\times\mathbb{R}^k$ (the product of an m-sphere by a k-dimensional Euclidean space). More generally, such a global Sobolev inequality requires too much "uniformity" of the Riemannian manifold M. Fortunately, there is a way to cope partially with this difficulty. The idea is to use *families of local Sobolev inequalities* instead of one global Sobolev inequality. For any ball $B = B(x, r)$ on a complete Riemannian manifold, one can find a constant $C(B)$ such that, for any smooth function f *with compact support in* B,

$$\left(\int_B |f|^q d\mu\right)^{2/q} \le \frac{C(B)r^2}{\mu(B)^{2/\nu}} \int_B \left(|\nabla f|^2 + r^{-2}|f|^2\right) d\mu$$

where $q, \nu > 2$ are some fixed constants related by $1/q = 1/2 - 1/\nu$. A lot of information is encoded in the behavior of the function $B \mapsto C(B)$. The simplest and perhaps most interesting case is when this function is bounded, that is, $\sup_B C(B) = C < \infty$. This can happen in cases where the global Sobolev inequality

$$\left(\int_M |f|^q d\mu\right)^{2/q} \le C \int_M \left(|\nabla f|^2\right) d\mu$$

does *not* hold. For instance, the manifold $\mathbb{S}^m \times \mathbb{R}^k$, $m+k > 2$ does not satisfy any global Sobolev inequality (assuming $m \neq 0$) but satisfies a family of local Sobolev inequalities with $\nu = m+k$, $q = 2\nu/(\nu-2)$ and $\sup_B C(B) = C < \infty$. In the other direction, the hyperbolic space of dimension n satisfies the same global Sobolev inequality as $\mathbb{R}^n$ but does not have $\sup_B C(B) < \infty$. In fact, as far as many applications are concerned (e.g., heat kernel bounds), a family of local Sobolev inequalities with $\sup_B C(B) < \infty$ contains more useful information than a global Sobolev inequality.

Chapter 5 develops these ideas and culminates with a complete proof of the following theorem, where $V(x, r)$ denotes the volume of the ball of center x and radius r, and d is the Riemannian distance. For any $x \in M$ and $s, r > 0$, let $Q = Q(x, s, r)$ be the time–space cylinder

$$Q(x, s, t) = (s - r^2, s) \times B(x, r).$$

Let Q_+, Q_- be respectively the upper and lower subcylinders

$$\begin{aligned} Q_+ &= (s - (1/4)r^2, s) \times B(x, (1/2)r) \\ Q_- &= (s - (3/4)r^2, s - (1/2)r^2) \times B(x, (1/2)r). \end{aligned}$$

We say that M satisfies the scale-invariant parabolic Harnack principle if there exists a constant C such that for any $x \in M$ and $s, r > 0$, and any positive solution u of $(\partial_t + \Delta)u = 0$ in $Q = Q(x, s, r)$, we have

$$\sup_{Q_-}\{u\} \le C \inf_{Q_+}\{u\}.$$

Theorem *A complete Riemannian manifold M satisfies the scale-invariant parabolic Harnack principle if and only if M satisfies the doubling property*

$$\forall x \in M, \quad \forall r > 0, \quad V(x, 2r) \le D_0 V(x, r)$$

and the scale-invariant Poincaré inequality

$$\forall B = B(x,r), \quad \int_B |f - f_B|^2 d\mu \le P_0 r^2 \int_B |\nabla f|^2 d\mu$$

where f_B denotes the mean of $f \in C^\infty(B)$ over the ball B.

In fact, the equivalent properties above are also equivalent to the fact that the heat kernel $h(t,x,y)$ satisfies the two-sided Gaussian estimate

$$\forall t > 0, \ \ \forall x, y \in M, \quad \frac{c_1 e^{-C_1 d(x,y)^2/t}}{V(x,\sqrt{t})} \le h(t,x,y) \le \frac{C_2 e^{-c_2 d(x,y)^2/t}}{V(x,\sqrt{t})}.$$

Such a two-sided heat kernel bound was first derived for uniformly elliptic divergence form second order differential operators in $\mathbb{R}^n$ by Aronson [3].

These results are taken from [32, 74] (a more complete discussion is given at the beginning of Section 5.5). The equivalence between the parabolic Harnack inequality on the one hand and the (more geometric) doubling property and Poincaré inequality on the other hand is a very useful tool. Both directions of this equivalence are interesting and this illustrated by a few simple examples. For instance, it follows from the theorem above that the parabolic Harnack principle is stable under quasi-isometries.

Chapter 1

Sobolev inequalities in $\mathbb{R}^n$

1.1 Sobolev inequalities

1.1.1 Introduction

How can one control the size of a function in terms of the size of its gradient? The well-known Sobolev inequalities answer precisely this question in multidimensional Euclidean spaces. On the real line, the answer is given by a simple yet extremely useful calculus inequality. Namely, for any smooth function f with compact support on the line,

$$|f(t)| \leq \frac{1}{2}\int_{-\infty}^{+\infty} |f'(s)|ds. \tag{1.1.1}$$

The factor 1/2 in this inequality comes from the fact that f vanishes at both $+\infty$ and $-\infty$. In this respect, note that if f is smooth but no other restriction is imposed the inequality above may fail.

It is natural to wonder if there is such an inequality for smooth compactly supported functions in higher-dimensional Euclidean spaces. More precisely, for each integer n, can one find $p, q > 0$ and $C > 0$ such that

$$\forall\, f \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \|f\|_q \leq C\|\nabla f\|_p? \tag{1.1.2}$$

Here and in the sequel $\mathcal{C}_0^\infty(\mathbb{R}^n)$ denotes the set of all smooth compactly supported functions in $\mathbb{R}^n$. For $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we set

$$\|f\|_q = \left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{1/q}, \quad \|f\|_\infty = \sup_{\mathbb{R}^n}\{|f|\}$$

and

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx\right)^{1/p}$$

where $\nabla f = (\partial_1 f, \ldots, \partial_n f)$ is the gradient of f and $|\nabla f| = \sqrt{\sum_1^n |\partial_i f|^2}$ is the Euclidean length of the gradient. In $\mathbb{R}^n$, we denote by $\mu_n = \mu$ the

Lebesgue measure and by μ_{n-1} the volume measure on smooth hypersurfaces of dimension $n-1$. When using coordinates $x = (x_1, \dots, x_n)$, we also write

$$d\mu(x) = dx = dx_1 \dots dx_n.$$

This question was first addressed in this form by Sobolev in [78] which appeared in Russian in 1938. Fixing a function $f \in C_0^\infty(\mathbb{R}^n)$ and replacing $x \mapsto f(x)$ by $x \mapsto f(tx)$, $t > 0$, in (1.1.2) yields

$$t^{-n/q}\|f\|_q \le C\, t^{1-n/p}\|\nabla f\|_p.$$

Letting t tend to zero and to infinity shows that (1.1.2) can only be satisfied if the exponents of t on both sides of the inequality above are the same. That is, (1.1.2) can only be satisfied if

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}, \text{ i.e., } q = \frac{np}{n-p}. \tag{1.1.3}$$

For instance, in $\mathbb{R}^2$, this says that one might possibly have

$$\forall f \in C_0^\infty(\mathbb{R}^2), \quad \|f\|_\infty \le \int_{\mathbb{R}^2} |\nabla f(y)|^2 dy. \tag{1.1.4}$$

The next example shows that this last inequality fails to be true.

EXAMPLE 1.1.1: Consider the function

$$f(x) = \begin{cases} \log|\log|x|| & \text{if } |x| \le 1/e \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\nabla f\|_2^2 = 2\pi \int_0^{1/e} \frac{dr}{r|\log r|^2} = 2\pi$ but f is not bounded. Of course, f is not smooth, but it can easily be approximated by smooth functions f_n such that $\|\nabla f_n\|_2 \to \|\nabla f\|_2$ and $f_n \to f$. This shows that that (1.1.4) cannot be true.

What is true is recorded in the following theorem.

Theorem 1.1.1 *Fix an integer $n \ge 2$ and a real p, $1 \le p < n$ and set $q = np/(n-p)$. Then there exists a constant $C = C(n,p)$ such that*

$$\forall f \in C_0^\infty(\mathbb{R}^n), \quad \|f\|_q \le C\|\nabla f\|_p. \tag{1.1.5}$$

This inequality is called the Sobolev inequality although the case $p = 1$ is not contained in [78]. Note that the case $p = n$ (i.e., $q = \infty$) is excluded in this result as should be the case according to the preceding example.

In the next few subsections we will give or outline several proofs of (1.1.5). As it turns out, when $p = 1$, (1.1.5) has a very simple proof based on

(1.1.1) and Hölder's inequality. This well-known proof (due independently to E. Gagliardo [28] and L. Nirenberg [68]) is presented in the next section. Moreover, as we shall see in 1.1.3 below, the case $p > 1$ follows from the case $p = 1$ by a simple trick.

We conclude this short introduction to Sobolev inequalities by recording a couple of useful remarks concerning the validity of (1.1.5). First, if (1.1.5) holds for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, it obviously also holds for a larger class of functions including for instance all $\mathcal{C}^1$ functions with compact support or even Lipschitz functions vanishing at infinity. In fact, (1.1.5) holds for all functions vanishing at infinity whose gradient in the sense of distributions is in L^p. Second, (1.1.5) restricted to non-negative functions in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ suffices to prove (1.1.5) in its full generality. Indeed, (1.1.5) for such functions implies that it also holds true for non-negative Lipschitz functions with compact support and, if $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $|f|$ is Lipschitz and satisfies $|\nabla|f|| \le |\nabla f|$ almost everywhere. It then follows that (1.1.5) holds for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

1.1.2 The proof due to Gagliardo and to Nirenberg

Recall that Hölder's inequality asserts that, for any positive measure μ,

$$\left|\int f g d\mu\right| \le \|f\|_p \|g\|_{p'}$$

for all $f \in L^p(\mu)$, $g \in L^{p'}(\mu)$, $1 \le p, p' \le \infty$ with $1 = 1/p + 1/p'$. By a simple induction we find that

$$\left|\int f_1 f_2 \dots f_k d\mu\right| \le \|f_1\|_{p_1} \|f_2\|_{p_2} \dots \|f_k\|_{p_k} \tag{1.1.6}$$

for all $f_i \in L^{p_i}$, $1 \le i \le k$, $1 \le p_i \le \infty$, $1/p_1 + 1/p_2 + \dots + 1/p_k = 1$.

Now, fix $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. By (1.1.1), for any $x = (x_1, \dots, x_n)$ and any integer $1 \le i \le n$, we have

$$|f(x)| \le \frac{1}{2} \int_{-\infty}^{+\infty} |\partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)| dt$$

(with the obvious interpretation if $i = 1$ or n). Set

$$F_i(x) = \int_{-\infty}^{+\infty} |\partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)| dt$$

and

$$F_{i,m}(x) = \begin{cases} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\partial_i f(x)| dx_1 \dots dx_m & \text{if } i \le m \\ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F_i(x) dx_1 \dots dx_m & \text{if } i > m. \end{cases}$$

Note that each F_i depends only on $n - 1$ variables, i.e., all coordinates but the i^{th}. Similarly, $F_{i,m}$ depends on either $n - m$ or $n - m - 1$ variables

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depending on whether $i \le m$ or $i > m$. In particular, for $m = n$, $F_{i,n}(x) = \int_{\mathbb{R}^n} |\partial_i f(y)| dy$ is a constant function. Now, we can estimate f by

$$|f| \le (1/2)(F_1 \dots F_n)^{1/n}$$

so that

$$|f|^{n/(n-1)} \le (1/2)^{n/(n-1)} (F_1 \dots F_n)^{1/(n-1)}.$$

Using (1.1.6) with $k = n-1$, $p_1 = p_2 = \cdots = p_k = n-1$ and induction on $m \le n$, one easily proves that

$$\int \cdots \int |f(x)|^{n/(n-1)} dx_1 \dots dx_m \le (1/2)^{n/(n-1)} (F_{1,m}(x) \dots F_{n,m}(x))^{1/(n-1)}.$$

For $m = n$ this reads

$$\|f\|_{n/(n-1)} \le (1/2) \left(\prod_1^n \|\partial_i f\|_1 \right)^{1/n}. \tag{1.1.7}$$

As $(\prod_1^n a_i)^{1/n} \le \frac{1}{n} \sum_1^n a_i$ for any positive numbers a_i and integer n, we obtain

$$\|f\|_{n/(n-1)} \le \frac{1}{2n} \sum_1^n \|\partial_i f(x)\|_1 dx \le \frac{1}{2\sqrt{n}} \|\nabla f\|_1. \tag{1.1.8}$$

To see the last inequality, use $\sum_1^n |\partial_i f| \le \sqrt{n} |\nabla f|$. This proves (1.1.5) for $p = 1$.

1.1.3 $p = 1$ implies $p \ge 1$

Assume that (1.1.5) holds for $p = 1$, that is,

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \|f\|_{n/(n-1)} \le C \|\nabla f\|_1. \tag{1.1.9}$$

Fix $p > 1$. For any $\alpha > 1$ and $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, note that $|f|^\alpha$ is $\mathcal{C}^1$, has compact support, and satisfies

$$|\nabla |f|^\alpha| = \alpha |f|^{\alpha-1} |\nabla f|.$$

Since we can easily approximate $|f|^\alpha$ by a sequence (f_i) of smooth functions with compact support such that $\nabla f_i \to \nabla |f|^\alpha$, inequality (1.1.9) holds with f replaced by $|f|^\alpha$. This yields

$$\begin{aligned} \|f\|_{\alpha n/(n-1)}^\alpha &\le C\alpha \int |f(x)|^{\alpha-1} |\nabla f(x)| dx \\ &\le C\alpha \left(\int |f(x)|^{(\alpha-1)p'} dx \right)^{1/p'} \left(\int |\nabla f(x)|^p dx \right)^{1/p} \end{aligned}$$