A Course in
Combinatorics
SECOND EDITION

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Graphs

A *graph* G consists of a set V (or $V(G)$) of *vertices*, a set E (or $E(G)$) of *edges*, and a mapping associating to each edge $e \in E(G)$ an unordered pair x, y of vertices called the *endpoints* (or simply the *ends*) of e. We say an edge is *incident* with its ends, and that it *joins* its ends. We allow $x = y$, in which case the edge is called a *loop*. A vertex is *isolated* when it is incident with no edges.

It is common to represent a graph by a *drawing* where we represent each vertex by a point in the plane, and represent edges by line segments or arcs joining some of the pairs of points. One can think e.g. of a network of roads between cities. A graph is called *planar* if it can be drawn in the plane such that no two edges (that is, the line segments or arcs representing the edges) cross. The topic of planarity will be dealt with in Chapter 33; we wish to deal with graphs more purely combinatorially for the present.

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

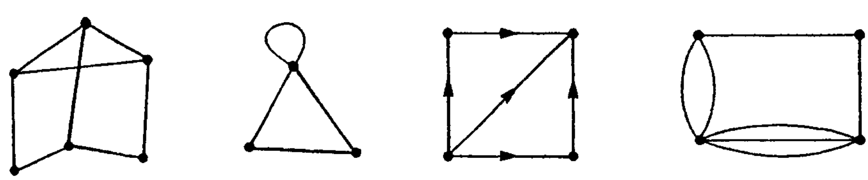
Figure 1.1

Thus a graph is described by a table such as the one in Fig. 1.1 that lists the ends of each edge. Here the graph we are describing has vertex set $V = \{x, y, z, w\}$ and edge set $E = \{a, b, c, d, e, f, g\}$; a drawing of this graph may be found as Fig. 1.2(iv).

A graph is *simple* when it has no loops and no two distinct edges have exactly the same pair of ends. Two nonloops are *parallel* when they have the same ends; graphs that contain them are called *multigraphs* by some authors, or are said to have 'multiple edges'.

If an *ordered* pair of vertices is associated to each edge, we have a *directed graph* or *digraph*. In a drawing of a digraph, we use an arrowhead to point from the first vertex (the *tail*) towards the second vertex (the *head*) incident with an edge. For a *simple* digraph, we disallow loops and require that no two distinct edges have the same ordered pair of ends.

When dealing with simple graphs, it is often convenient to identify the edges with the unordered pairs of vertices they join; thus an edge joining x and y can be called $\{x, y\}$. Similarly, the edges of a simple digraph can be identified with ordered pairs (x, y) of distinct vertices.



(i) graph (ii) graph with loop (iii) digraph (iv) multiple edges

Figure 1.2

There are several ways to draw the same graph. For example, the two graphs of Fig. 1.3 are essentially the same.

We make this more precise, but to avoid unnecessarily technical definitions at this point, let us assume that all graphs are undirected and simple for the next two definitions.

We say two graphs are *isomorphic* if there is a one-to-one correspondence between the vertex sets such that if two vertices are joined by an edge in one graph, then the corresponding vertices are joined by an edge in the other graph. To show that the two graphs in Fig. 1.3 are the same, find a suitable numbering of the vertices in both graphs (using $1, 2, 3, 4, 5, 6$) and observe that the edge sets are the same sets of unordered pairs.

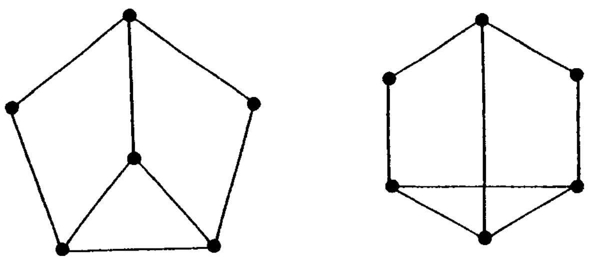


Figure 1.3

A permutation σ of the vertex set of a graph G with the property that $\{a, b\}$ is an edge if and only if $\{\sigma(a), \sigma(b)\}$ is an edge, is called an *automorphism* of G.

Problem 1A. (i) Show that the drawings in Fig. 1.4 represent the same graph (or isomorphic graphs).

(ii) Find the group of automorphisms of the graph in Fig. 1.4. Remark: There is no quick or easy way to do this unless you are lucky; you will have to experiment and try things.

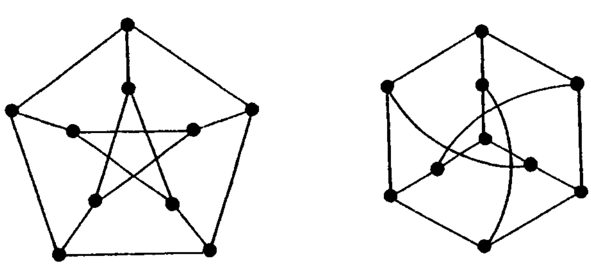


Figure 1.4

The *complete* graph K_n on n vertices is the simple graph that has all $\binom{n}{2}$ possible edges.

Two vertices a and b of a graph G are called *adjacent* if they are distinct and joined by an edge. We will use $\Gamma(x)$ to denote the set of all vertices adjacent to a given vertex x; these vertices are also called the *neighbors* of x.

The number of edges incident with a vertex x is called the *degree* or the *valency* of x. Loops are considered to contribute 2 to the valency, as the pictures we draw suggest. If all the vertices of a graph have the same degree, then the graph is called *regular.*

One of the important tools in combinatorics is the method of *counting* certain objects in two different ways. It is a well known fact that if one makes no mistakes, then the two answers are the same. We give a first elementary example. A graph is *finite* when both $E(G)$ and $V(G)$ are finite sets. We will be primarily concerned with finite graphs, so much so that it is possible we have occasionally forgotten to specify this condition as a hypothesis in some assertions.

Theorem 1.1. *A finite graph G has an even number of vertices with odd valency.*

PROOF: Consider a table listing the ends of the edges, as in Fig. 1.1. The number of entries in the right column of the table is twice the number of edges. On the other hand, the degree of a vertex x is, by definition, the number of times it occurs in the table. So the number of entries in the right column is

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|. \tag{1.1}$$

The assertion follows immediately. □

The equation (1.1) is simple but important. It might be called the 'first theorem of graph theory', and our Theorem 1.1 is its first corollary.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and the ends of an edge $e \in E(H)$ are the same as its ends in G. H is a *spanning* subgraph when $V(H) = V(G)$. The subgraph of G *induced* by a subset S of vertices of G is the subgraph whose vertex set is S and whose edges are *all* the edges of G with both ends in S.

A *walk* in a graph G consists of an alternating sequence

$$x_0, e_1, x_1, e_2, x_2, \dots, x_{k-1}, e_k, x_k$$

of vertices x_i, not necessarily distinct, and edges e_i so that the ends of e_i are exactly x_{i-1} and x_i, $i = 1, 2, \dots, k$. Such a walk has *length* k. If the graph is simple, a walk is determined by its sequence of vertices, any two successive elements of which are adjacent.

If the edge terms $e_1, \dots, e_k$ are distinct, then the walk is called a *path* from x_0 to x_k. If $x_0 = x_k$, then a walk (or path) is called *closed.* A *simple* path is one in which the vertex terms $x_0, x_1, \dots, x_k$ are also distinct, although we say we have a *simple closed path* when $k \geq 1$ and all vertex terms are distinct except $x_0 = x_k$.

If a path from x to y exists for every pair of vertices x, y of G, then G is called *connected.* Otherwise G consists of a number of connected *components* (maximal connected subgraphs). It will be convenient to agree that the null graph with no vertices and no edges is not connected.

Problem 1B. Suppose G is a simple graph on 10 vertices that is not connected. Prove that G has at most 36 edges. Can equality occur?

The length of the shortest walk from a to b, if such walks exist, is called the *distance* $d(a, b)$ between these vertices. Such a shortest walk is necessarily a simple path.

Example 1.1. A well known graph has the mathematicians of the world as vertices. Two vertices are adjacent if and only if they have published a joint paper. The distance in this graph from some mathematician to the vertex P. Erdős is known as his or her Erdős-number.

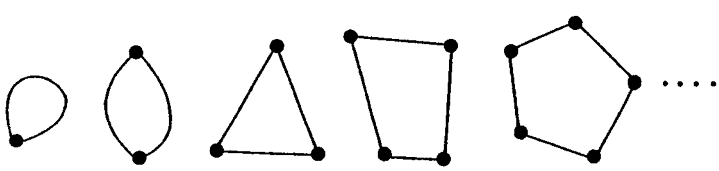


Figure 1.5

A *polygon* is the 'graph of' a simple closed path, but more precisely it can be defined as a finite connected graph that is regular of degree 2. There is, up to isomorphism, exactly one polygon P_n

with n vertices (often called the *n-gon*) for each positive integer n. The sequence of polygons is shown in Fig. 1.5.

A connected graph that contains no simple closed paths, i.e. that has no polygons as subgraphs, is called a *tree*.

Problem 1C. Show that a connected graph on n vertices is a tree if and only if it has $n-1$ edges.

Problem 1D. The *complete bipartite graph* $K_{n,m}$ has $n+m$ vertices $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, and as edges all mn pairs $\{a_i, b_j\}$. Show that $K_{3,3}$ is not planar.

No introduction to graph theory can omit the problem of the bridges of Königsberg (formerly a city in Prussia). The river Pregel flowed through this city and split into two parts. In the river was the island Kneiphof. There were seven bridges connecting different parts of the city as shown in the diagram of Fig. 1.6.

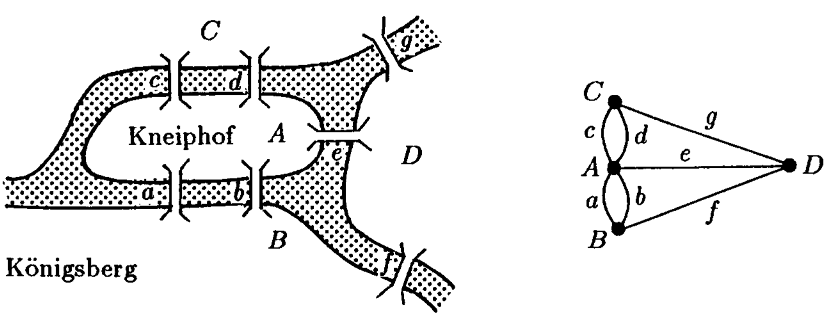


Figure 1.6

In a paper written in 1736 by L. Euler (considered the first paper on graph theory) the author claims that the following question was considered difficult: Is it possible to make a walk through the city, returning to the starting point and crossing each bridge exactly once? This paper has led to the following definition. A closed path through a graph using every edge once is called an *Eulerian circuit* and a graph that has such a path is called an *Eulerian graph*.

Theorem 1.2. *A finite graph G with no isolated vertices (but possibly with multiple edges) is Eulerian if and only if it is connected and every vertex has even degree.*

PROOF: That G must be connected is obvious. Since the path enters a vertex through some edge and leaves by another edge, it is clear that all degrees must be even. To show that the conditions are sufficient, we start in a vertex x and begin making a path. We keep going, never using the same edge twice, until we cannot go further. Since every vertex has even degree, this can only happen when we return to x and all edges from x have been used. If there are unused edges, then we consider the subgraph formed by these edges. We use the same procedure on a component of this subgraph, producing a second closed path. If we start this second path in a point occurring in the first path, then the two paths can be combined to a longer closed path from x to x. Therefore the longest of these paths uses all the edges. □

The problem of the bridges of Königsberg is described by the graph in Fig. 1.6. No vertex has even degree, so there is no Eulerian circuit.

One can consider a similar problem for digraphs. The necessary and sufficient condition for a directed Eulerian circuit is that the graph is connected and that each vertex has the same 'in-degree' as 'out-degree'.

Example 1.2. A puzzle with the name *Instant Insanity* concerns four cubes with faces colored red, blue, green, and yellow, in such a way that each cube has at least one face of each color. The problem is to make a stack of these cubes so that all four colors appear on each of the four sides of the stack. In Fig. 1.7 we describe four possible cubes in flattened form.

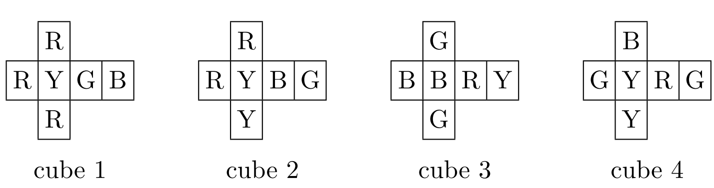
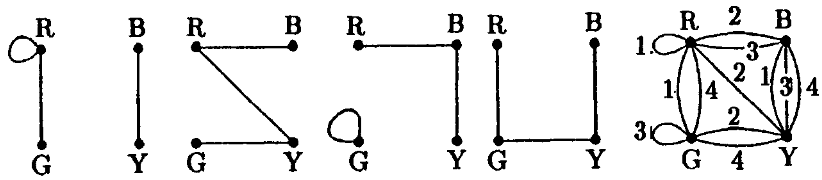


Figure 1.7

It is not a very good idea to try all possibilities. A systematic approach is as follows. The essential information about the cubes is given by the four graphs in Fig. 1.8.



cube 1 cube 2 cube 3 cube 4 the graph G

Figure 1.8

An edge indicates that the two adjacent colors occur on opposite faces of the cube. We obtain a graph G by superposition of the four graphs and number the edges according to their origin. It is not difficult to see that we need to find in G two subgraphs that are regular of degree 2, with edges numbered $1, 2, 3, 4$ and such that they have no edge in common. One of the subgraphs tells us which pairs of colors to align on the left side and right side of the stack. The other graph describes the colors on front and back. Of course it is easy to rotate the cubes in such a way that the colors are where we wish them to be. The point of the example is that it takes only a minute to find two subgraphs as described above. In this example the solution is unique.

We mention a concept that seems similar to Eulerian circuits but that is in reality quite different. A *Hamiltonian circuit* in a graph G is a simple closed path that passes through each *vertex* exactly once (rather than each *edge*). So a graph admits a Hamiltonian circuit if and only if it has a polygon as a spanning subgraph. In the mid-19th century, Sir William Rowan Hamilton tried to popularize the exercise of finding such a closed path in the graph of the dodecahedron (Fig. 1.9).

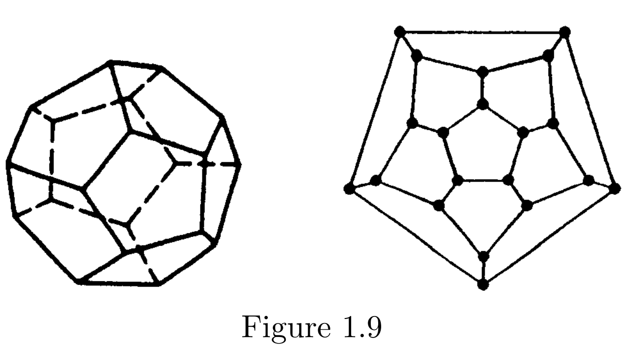


Figure 1.9

The graph in Fig. 1.4 is called the Petersen graph (cf. Chapter 21) and one of the reasons it is famous is that it is *not* 'Hamiltonian'; it contains n-gons only for $n = 5, 6, 8, 9$, and not when $n = 7$ or $n = 10$.

By Theorem 1.2, it is easy to decide whether a graph admits an Eulerian circuit. A computer can easily be programmed to check whether the degrees of a graph are even and whether the graph is connected, and even to produce an Eulerian circuit when one exists. In contrast to this, the problem of deciding whether an arbitrary graph admits a Hamiltonian circuit is likely 'intractable'. To be more precise, it has been proved to be *NP-complete*—see Garey and Johnson (1979).

Problem 1E. Let $A_1, \dots, A_n$ be n distinct subsets of the n-set $N := \{1, \dots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \backslash \{x\}$, $1 \le i \le n$, are all distinct. To do this, form a graph G on the vertices A_i with an edge with 'color' x between A_i and A_j if and only if the symmetric difference of the sets A_i and A_j is $\{x\}$. Consider the colors occurring on the edges of a polygon. Show that one can delete edges from G in such a way that *no* polygons are left and the number of different colors remains the same. Then use 1C. (This idea is due to J. A. Bondy (1972).)

Problem 1F. The *girth* of a graph is the length of the smallest polygon in the graph. Let G be a graph with girth 5 for which all vertices have degree $\ge d$. Show that G has at least $d^2 + 1$ vertices. Can equality hold?

Problem 1G. Show that a finite simple graph has at least two vertices with the same degree.

Problem 1H. A graph on the vertex set $\{1, 2, \dots, n\}$ is often described by a matrix A of size n, where a_{ij} and a_{ji} are equal to the number of edges with ends i and j. What is the combinatorial interpretation of the entries of the matrix A^2?

Problem 1I. Let $Q := \{1, 2, \dots, q\}$. Let G be a graph with the elements of Q^n as vertices and an edge between $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ if and only if $a_i \ne b_i$ for exactly one value of i. Show that G is Hamiltonian.

Problem 1J. Let G be a simple graph on n vertices ($n > 3$) with no vertex of degree $n-1$. Suppose that for any two vertices of G, there is a *unique* vertex joined to both of them.

(i) If x and y are not adjacent, prove that they have the same degree.

(ii) Now show that G is a regular graph.

Notes.

Paul Erdős (1913–1996) (cf. Example 1.1) was probably the most prolific mathematician of the 20th century with well over 1400 papers having been published. His contributions to combinatorics, number theory, set theory, etc., include many important results. He collaborated with many mathematicians all over the world, all of them proud to have Erdős-number 1, among them the authors of this book; see J. W. Grossman (1997).

Leonhard Euler (1707–1783) was a Swiss mathematician who spent most of his life in St. Petersburg. He was probably the most productive mathematician of all times. Even after becoming blind in 1766, his work continued at the same pace. The celebration in 1986 of the 250th birthday of graph theory was based on Euler's paper on the Königsberg bridge problem. Königsberg is now the city of Kaliningrad in Russia.

For an elementary introduction to graph theory, we recommend R. J. Wilson (1979), and J. J. Watkins and R. J. Wilson (1990).

Sir William Rowan Hamilton (1805–1865) was an Irish mathematician. He was considered a genius. He knew 13 languages at the age of 12 and was appointed professor of astronomy at Trinity College Dublin at the age of 22 (before completing his degree). His most important work was in mathematical physics.

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