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Pertti Lounesto

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Pertti Lounesto

Helsinki Polytechnic Stadia



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Contents

Preface	vii
Mathematical Notation	ix
1. Vectors and linear spaces	1
2. Complex numbers	18
3. Bivectors and the exterior algebra	33
4. Pauli spin matrices and spinors	50
5. Quaternions	67
6. The fourth dimension	80
7. The cross product	92
8. Electromagnetism	100
9. Lorentz transformations	118
10. The Dirac equation	135
11. Fierz identities and boomerangs	152
12. Flags, poles and dipoles	162
13. Tilt to the opposite metric	174
14. Definitions of the Clifford algebra	188
15. Witt rings and Brauer groups	195
16. Matrix representations and periodicity of 8	205
17. Spin groups and spinor spaces	219
18. Scalar products of spinors and the chessboard	231
19. Möbius transformations and Vahlen matrices	244
20. Hypercomplex analysis	255
21. Binary index sets and Walsh functions	279
22. Chevalley's construction and characteristic 2	288
23. Octonions and triality	300
A history of Clifford algebras	320
Selected reading	331
Index	335

Preface

This book is intended for people who are not primarily algebraists, but nonetheless get involved in the subject through other areas – having backgrounds, in say, engineering, physics, geometry or analysis. Readers of this book may have different starting levels, backgrounds and goals.

Chapters 1–2 form a unit for an undergraduate course: Vectors, the scalar product; complex numbers, a geometrical interpretation of the imaginary unit $i = \sqrt{-1}$.

Chapters 3–7 guide the reader through bottlenecks and provide necessary building blocks: Bivectors and the exterior product. Pauli spin matrices and Pauli spinors. Quaternions and the fourth dimension. The cross product is generalized to higher dimensions.

Chapters 8–10 aim to serve readers with different backgrounds: Electromagnetism, special relativity, Dirac theory. The Dirac equation is formulated with complex column spinors, spinors in minimal left ideals and considering spinors as operators.

Chapters 11–13 discuss physical applications of spinors: In the case of an electron, Fierz identities are sufficient to reconstruct spinors from their physical observables, but this is not the case for the neutrino. Boomerangs are introduced to handle neutrinos. A new class of spinors is identified by its bilinear observables: the *flag-dipole spinors* which reside between Weyl, Majorana and Dirac spinors.

Chapters 14–15 are more algebraic than the previous chapters. Clifford algebras are defined for the first time. Finite fields. Isometry classes of quadratic forms and their Witt rings. Tensor products of algebras and Brauer groups are discussed.

Chapters 16–19 view Clifford algebras through matrix algebras: Clifford algebras are given isomorphic images as matrix algebras, Cartan's periodicity of 8, spin groups and their matrix images in lower dimensions, scalar products of spinors with a chessboard of their automorphism groups, Möbius transformations represented by Vahlen matrices.

Chapters 20–23 discuss miscellaneous mathematical topics. A one-variable higher-dimensional generalization of complex analysis: Cauchy's integral formula is generalized to higher dimensions. Multiplication rule of standard basis elements of a Clifford algebra and its relation to Walsh functions. Multivector structure of Clifford algebras and the

non-existence of k -vectors, $k \geq 2$, in characteristic 2. In the last chapter, we come into contact with final frontiers science: an exceptional phenomenon in dimension 8, triality, which has no counterpart in any other dimension.

The first parts of initial chapters are accessible without knowledge of other parts of the book – thus a teacher may choose his own path for his lectures on Clifford algebras. The latter parts of the chapters are sometimes more advanced, and can be left as independent study for interested students.

Introduction of the Clifford algebra of multivectors and spinors can be motivated in two different ways, in physics and in geometry:

- (i) In physics, the concept of Clifford algebra, as such or in a disguise, is a necessity in the description of electron spin: Spinors cannot be constructed by tensorial methods, in terms of exterior powers of the vector space.
- (ii) In geometry, information about orientation of subspaces can be encoded in simple multivectors, which can be added and multiplied. Physicists are familiar with this tool in the special case of one-dimensional subspaces of oriented line-segments, which they manipulate by vectors (not by projection operators, which lose information about orientation).

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Helsinki, November 2000

PERTTI LOUNESTO

Mathematical Notation

V	linear space
\mathbb{F}	field
\mathbb{D}	division ring (typically \mathbb{R} , \mathbb{C} , \mathbb{H})
\mathbb{H}	division ring of quaternions
\mathbb{O}	division algebra of octonions
$\text{Mat}(d, \mathbb{F})$	$d \times d$ matrix algebra over \mathbb{F}
$\text{Mat}(d, \mathbb{D})$	matrix algebra over $\text{Cen}(\mathbb{D})$ with entries in \mathbb{D}
${}^2\mathbb{F}$	double ring $\mathbb{F} \times \mathbb{F}$ of the field \mathbb{F}
${}^2\text{Mat}(d, \mathbb{F})$	direct sum $\text{Mat}(d, \mathbb{F}) \oplus \text{Mat}(d, \mathbb{F}) \simeq \text{Mat}(d, {}^2\mathbb{F})$
A	an algebra
$\text{Cen}(A)$	the center of an algebra A
\mathbb{A}	\mathbb{R} , \mathbb{C} , \mathbb{H} , ${}^2\mathbb{R}$, or ${}^2\mathbb{H}$
\mathbb{R}^n	n -dimensional real linear space
$\mathbb{R}^n = \mathbb{R}^{n,0}$	n -dimensional Euclidean space
\mathbb{C}^n	n -dimensional complex linear space
\mathbb{H}^n	n -dimensional module over \mathbb{H}
$\mathbb{R}^{p,q}$	real quadratic space (p for positive and q for negative)
$\mathcal{C}\ell_{p,q}$	Clifford algebra of $\mathbb{R}^{p,q}$
Q	quadratic form
$\mathcal{C}\ell(Q)$	Clifford algebra of the quadratic form Q ; $\mathbf{x}^2 = Q(\mathbf{x})$
$\mathcal{C}\ell_3 \simeq \text{Mat}(2, \mathbb{C})$	Clifford algebra of the Euclidean space \mathbb{R}^3
$\mathcal{C}\ell_n = \mathcal{C}\ell_{n,0}$	Clifford algebra of the Euclidean space $\mathbb{R}^n = \mathbb{R}^{n,0}$
$\mathcal{C}\ell_{1,3} \simeq \text{Mat}(2, \mathbb{H})$	Clifford algebra of the Minkowski space $\mathbb{R}^{1,3}$
$\mathcal{C}\ell_{3,1} \simeq \text{Mat}(4, \mathbb{R})$	Clifford algebra of the Minkowski space $\mathbb{R}^{3,1}$
\hat{u}	grade involute of $u \in \mathcal{C}\ell(Q) = \mathcal{C}\ell^+(Q) \oplus \mathcal{C}\ell^-(Q)$
\tilde{u}	reverse of $u \in \mathcal{C}\ell(Q)$; $\tilde{\mathbf{x}} = \mathbf{x}$ for $\mathbf{x} \in V$
\bar{u}	Clifford-conjugate of $u \in \mathcal{C}\ell(Q)$; $\bar{\mathbf{x}} = -\mathbf{x}$ for $\mathbf{x} \in V$
u^*	complex conjugate of u (in chapter 2 also \bar{u})
f	primitive idempotent of $\mathcal{C}\ell_{p,q}$
$S = \mathcal{C}\ell_{p,q}f$	spinor space (minimal left ideal of $\mathcal{C}\ell_{p,q}$)
\check{S}, \check{U}	left ideal $S \oplus \hat{S}$ or companion of $U \in SO(8)$
\mathbb{Z}_n	$\{0, 1, \dots, n-1\}$ or $\{1, e^{i2\pi/n}, \dots, e^{i2\pi(n-1)/n}\}$