

Cambridge University Press

978-0-521-00419-0 - The Eightfold Way: The Beauty of Klein's Quartic Curve

Edited by Silvio Levy

Frontmatter

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Felix Klein discovered in the 1870s that the simple equation $x^3y + y^3z + z^3x = 0$ (in complex projective coordinates) describes a surface having many remarkable properties, including 336-fold symmetry – the maximum possible for any surface of this genus. Since then this object has come up in different guises in several areas of mathematics.

The mathematical sculptor Helaman Ferguson has tried to distill some of the beauty and remarkable properties of this surface in the form of a sculpture that he entitled *The Eightfold Way*, permanently installed at the Mathematical Sciences Research Institute in Berkeley.

This volume seeks to explore the rich tangle of properties and theories surrounding this object, as well as its esthetic aspects. It contains:

- The text written by William Thurston to explain the sculpture to a wide public at the time of its inauguration.
- A broad overview of the position of the Klein quartic in mathematics, with articles by Hermann Karcher and Matthias Weber (geometry), Noam Elkies (number theory), and A. Murray Macbeath (Riemann surfaces).
- A historical overview by Jeremy Gray.
- A richly illustrated essay by the sculptor, Helaman Ferguson.
- An exploration of related curves by Allan Adler, with new results and exposition of old ones.
- The first English translation of Klein's seminal article, "On the order-seven transformation of elliptic functions."

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35

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Contents

Preface: MSRI and the Klein Quartic	ix
The Eightfold Way: A Mathematical Sculpture by Helaman Ferguson WILLIAM P. THURSTON	1
The Geometry of Klein's Riemann Surface HERMANN KARCHER AND MATTHIAS WEBER	9
The Klein Quartic in Number Theory NOAM ELKIES	51
Hurwitz Groups and Surfaces A. MURRAY MACBEATH	103
From the History of a Simple Group JEREMY GRAY	115
Eightfold Way: The Sculpture HELAMAN AND CLAIRE FERGUSON	133
Invariants of $SL_2(\mathbb{F}_q) \cdot \text{Aut}(\mathbb{F}_q)$ Acting on \mathbb{C}^n for $q = 2n \pm 1$ ALLAN ADLER	175
Hirzebruch's Curves $F_1, F_2, F_4, F_{14}, F_{28}$ for $\mathbb{Q}(\sqrt{7})$ ALLAN ADLER	221
On the Order-Seven Transformation of Elliptic Functions FELIX KLEIN	287

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MSRI and the Klein Quartic

On November 14, 1993, a marble and serpentine sculpture was unveiled at the Mathematical Sciences Research Institute in Berkeley, an event that marked one of the ways in which MSRI has been reaching out beyond its traditional role. The work had been commissioned from the famous mathematical sculptor Helaman Ferguson, thanks to a generous donation from Mitsubishi Electric Research Laboratories (MERL) made for the purpose. This sculpture, and the mathematical object that lies behind it, are the subject of this book.

Felix Klein discovered in 1878 that a certain surface, whose equation (in complex projective coordinates) he gave very simply as $x^3y + y^3z + z^3x = 0$, has a number of remarkable properties, including an incredible 336-fold symmetry. He arrived at it as a quotient of the upper complex half-plane by a modular group—the group of fractional linear transformations whose coefficients are integers and that reduce to the identity modulo 7. Since then, the same structure has come up in different guises in many areas of mathematics.

Ferguson's sculpture, *The Eightfold Way*, is a distillation of the beauty and remarkable properties of the Klein quartic. (See Plate 1 following page 142.) At the base is a two-color stone mosaic, representing the uniformization of the surface: a regular hyperbolic tessellation shown in the Poincaré model. Rising out of the central tile, a seven-sided black column cups the artist's Carrara marble rendition of the surface, which highlights its tetrahedral symmetry. The name *The Eightfold Way* is explained by the ridges and grooves that crisscross the otherwise smooth hand-polished surface: they represent the same tessellation, after the surface has folded over itself. If you run your finger along these curves, alternating left and right turns at each corner, you always come back to the beginning after eight turns. In the words of Claire Ferguson, the overall effect is that of "... a symphony of elegant counterpoint—as if Gothic tracery and Alhambra tilings were united in one work."

* * *

This book was a long time in the making, and I owe a debt of gratitude to all the contributors, both for their writing and for their good-humored cooperation during the often hectic process of proof review. I particularly want to thank Hermann Karcher and Matthias Weber, who contributed very early, for their

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MSRI AND THE KLEIN QUARTIC

patience; Helaman and Claire Ferguson, for supplying slides and for not biting my head off when I proceeded to lose them; Murray Macbeath, for responding promptly to a late request for a contribution; Jeremy Gray, for allowing me to reprint his *Intelligencer* article and for invaluable advice on the Klein translation; Noam Elkies and Allan Adler, for their thoroughness; and Bill Thurston, for putting all of this into motion. Thanks also to Lauren Cowles and Catherine Felgar, respectively Mathematics Editor and Production Editor at Cambridge University Press, for getting the book out in record time.

Silvio Levy

Berkeley, summer and fall 1998