The Theory of Composites

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1

Introduction

1.1. What are composites, and why study them?

Composites are prevalent in both nature and among engineered materials. Common metals are composites. When one breaks a rod of metal the polycrystalline nature becomes evident in the roughness of the surface of the break. The American Museum of Natural History in New York has a wonderful meteorite collection. Some of the polished cross sections through these meteorites clearly show the individual metal crystals. Martensite, which is typical of a shape memory material, has a laminar-type structure comprised of alternating layers of the two variants of martensite. Some rocks, such as sandstone, are aggregates of grains; other rocks, such as granite, are aggregates of crystals. In porous rocks the pores are often filled with a fluid such as salt water or oil. The study of composites in a geological context is important to the oil industry and for the study of earthquakes. Construction materials such as wood and concrete are composites. Bone is a porous composite. Fiberglass and lightweight carbon fiber composites have found applications ranging from the aerospace industry to sports equipment.

Colloidal suspensions, emulsions, foams, slurries, and clays are all examples of composites. Clouds, fog, mist, and rain are composites of air and water. High-altitude clouds are composites of air and ice crystals. Suspensions of volcanic dust in the upper atmosphere are known to significantly perturb temperatures around the earth. Air itself is an inhomogeneous medium with fluctuations in density that cause the twinkling of stars. Sea ice is a composite of ice and brine pockets, and modeling of its properties is important in global climate prediction. Wool and cotton are composites of fiber and air. Ceramics are composites. Solid rocket propellant is a composite of aluminum particles in an oxidizing matrix. Even chocolate chip ice cream is a composite. Basically, composites are materials that have inhomogeneities on length scales that are much larger than the atomic scale (which allows us to use the equations of classical physics at the length scales of the inhomogeneities) but which are essentially (statistically) homogeneous at macroscopic length scales, or least at some intermediate length scales. An alloy, having disorder on the atomic scale, is excluded from consideration (except if it is one of the phases in a larger composite, in which case it is treated as a homogeneous material). The book of Matthews and Rawlings (1994) gives many examples of natural and man-made composites.

Why do we study composites? One obvious answer is their usefulness, which will be discussed in the next section. A second, equally important reason is that what we learn from the field of composites could have far-reaching implications in many fields of science. Significant progress in improving our understanding of how microscopic behavior influences macroscopic behavior could impact our understanding of turbulence, of phase transitions involving many length scales, of how quantum behavior influences behavior on classical length scales, or, at the more extreme level, of how behavior on the Planck length scale, 10^{-33}cm, influences behavior on the atomic scale, 10^{-8}cm. While that may seem unlikely, it is hard to deny the impact that our understanding of classical physics had on the development of quantum mechanics. Therefore it is conceivable that a better understanding of classical questions involving multiple length scales could have large reverberations. A third compelling reason for studying composites is simply that there are many beautiful mathematical questions begging for answers. The solutions of some questions have already led to the development of new mathematical tools, and one can expect that the solution of the more challenging outstanding questions will open new mathematical frontiers.

The study of composites is a subject with a long history, which has attracted the interest of some of the greatest scientists. For example, Poisson (1826) constructed a theory of induced magnetism in which the body was assumed to be composed of conducting spheres embedded in a nonconducting material. Faraday (1839) proposed a model for dielectric materials that consisted of metallic globules separated by insulating material. Maxwell (1873) solved for the conductivity of a dilute suspension of conducting spheres in a conducting matrix. Rayleigh (1892) found a system of linear equations which, when solved, would give the effective conductivity of nondilute square arrays of cylinders or cubic lattices of spheres. Einstein (1905) calculated the effective shear viscosity of a suspension of rigid spheres in a fluid. The main historical developments are summarized in the articles of Landauer (1978) and Markov (2000).

1.2. What makes composites useful?

What gives composites their utility is that they often combine the attributes of the constituent materials. For example, suppose that one is given two isotropic conducting materials: a metal with high conductivity, and a plastic that is electrically insulating. If one places these two materials in alternating layers in a laminate, one obtains a highly anisotropic composite that has the conducting properties of the metal in directions parallel to the layers and the insulating properties of the plastic normal to the layers. Concrete is cheap and relatively light, but it breaks apart easily under tension. By contrast, steel is strong but expensive and heavy. By pouring the concrete around prestressed metal bars one obtains a composite, namely, reinforced concrete, that is cheap, relatively light, and strong. Wood is an example of a material that is strong in the fiber direction, but the fibers pull apart easily. By alternating layers of wood that are strong in the x_1 direction with layers of wood that are strong in the x_2 direction, one obtains a plywood that is strong in two directions, that is, in the (x_1, x_2)-plane.

By combining a compliant isotropic material that has low bulk and shear moduli with a stiff isotropic material that has high bulk and shear moduli, one can (with a judicious choice of microstructure) produce an elastically isotropic composite that effectively has the bulk modulus of the compliant phase and the shear modulus of the stiff phase. Such low-bulk and high-shear moduli materials are called negative Poisson's ratio materials: A rod of the material will expand laterally when stretched longitudinally. It was long a question of debate as to whether such materials could actually exist. Now their existence has been confirmed both experimentally and theoretically [see, for example, Lakes (1987) and Milton (1992) and references therein, and also section 30.5 on page 652]. A very simple two-dimensional microstructure that expands laterally when stretched longitudinally was designed and fabricated by Larsen, Sigmund, and Bouwstra (1997) and is illustrated in figure 1.1 on the facing page.

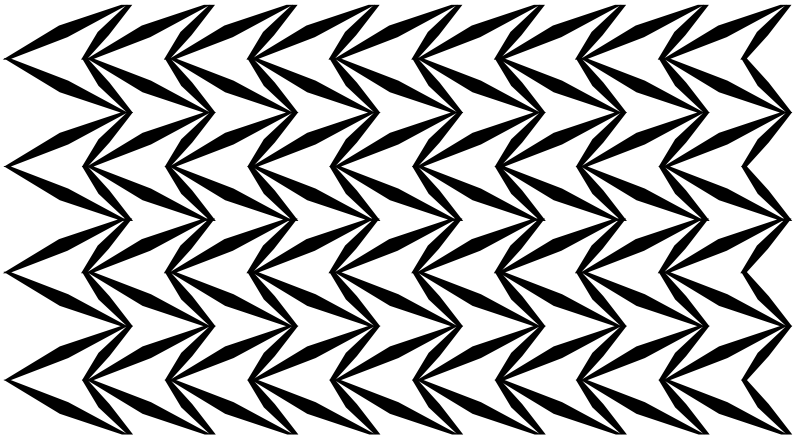
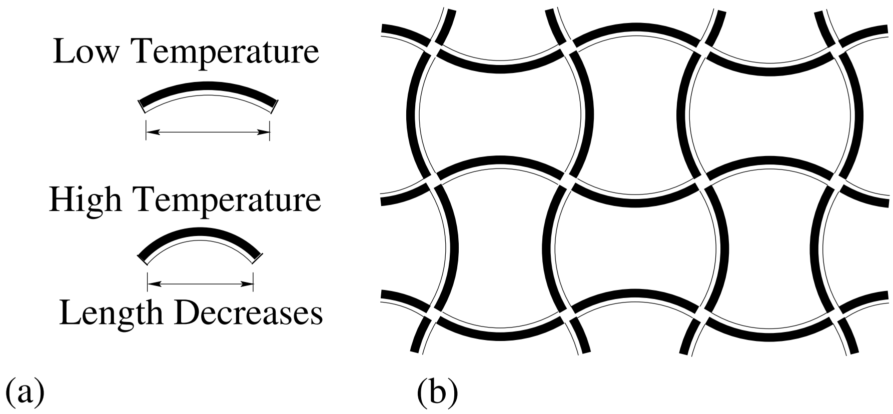


Figure 1.1. The two-dimensional microstructure of Larsen, Sigmund, and Bouwstra (1997), which will expand laterally when stretched longitudinally. Here the black region is relatively stiff and is surrounded by a void or very compliant material.

Sometimes the properties of a composite can be strikingly different from the properties of the constituent materials. To see this you can do the following experiment at home. Take a wine glass filled with air and strike it on the top (but not too hard) with a knife. It rings clearly. The same is true when it is filled with water. But add powdered Alka-Seltzer (or anything else that makes lots of bubbles) to the water and one just hears a thud. The acoustic properties of the bubbly fluid are quite different from those of either the air or the water. One application of this is to use a screen of bubbles to mask the engine or propeller noise of a submarine; as the oscillatory pressure in the sound wave compresses and decompresses each bubble, the water near each bubble is sheared, which dissipates the energy of the sound. The shear viscosity of the water is converted to the bulk viscosity of the bubbly fluid (Taylor 1954; see also section 11.4 on page 233). As another example, the beautiful red glass that one sees in some old church windows is a suspension of small gold particles in glass. The color arises not from any chemical effect but rather from the effective complex dielectric constant of the suspension at optical frequencies (Maxwell Garnett 1904). Opals consist of submicron spherical particles of silica arranged in a face-centered cubic array, diffracting light to create the brilliant colors that we see (Sanders 1964; Greer 1969). Similarly, the spines of a particular sea worm display a wonderful iridescence caused by light diffracting off a hexagonal array of fibers within each spine (Parker, McPhedran, McKenzie, Botten, and Nicorovici 2001).

By combining materials with positive thermal expansion coefficients it is possible to get a composite with a negative thermal expansion coefficient (Lakes 1996; Sigmund and Torquato 1996, 1997). This is most easily seen in a two-dimensional context. Following Lakes (1996) consider the structure of figure 1.2(b), where the cell walls consist of thin, stiff, curved metal strips with a low thermal expansion coefficient coated on the outside with a thick compliant strip of material with a high thermal expansion coefficient. As the composite is heated the strips become more tightly curved, as illustrated in figure 1.2(a) and consequently the material contracts, that is, it has a negative thermal expansion coefficient. Lakes also shows that it is possible to construct a porous composite with a significantly larger thermal expansion coefficient than either of the two phases. Bergman and Fel (1999) have shown that the thermoelectric power factor in a two-phase composite can be greater than the thermoelectric power factors of both phases.

A composite of a piezoelectric material and an elastic material can have a dramatically larger electrical response to hydrostatic compression than either phase alone. For example, when a cylindrical rod of the piezoelectric material lead zirconate-titanate (PZT) is compressed longitudinally, an electric field is generated parallel to the axis of the rod. When it is compressed transversely an electric field is generated in the opposite direction. If the rod is immersed in a fluid and hydrostatic pressure is applied, then the rod is compressed both axially and transversely and the induced electric fields almost cancel out. This canceling is avoided if an array of such rods is embedded in a polymer matrix, which restricts the amount of transverse compression. The electric field generated in the composite can be measured and thus small variations in the fluid pressure can be detected. Such composites are useful in the design of piezoelectric hydrophones for detecting low-frequency underwater acoustic waves (Klicker, Biggers, and Newnham 1981; Avellaneda and Swart 1998). The performance is greatly enhanced by using a negative Poisson's ratio material in place of the polymer (Smith 1991; Gibiansky and Torquato 1997; Avellaneda and Swart 1998; Sigmund, Torquato, and Aksay 1998).



(a) (b)

Figure 1.2. Two materials with positive thermal expansion coefficients can be combined in a porous structure to give a composite with a negative thermal expansion coefficient. The key observation, illustrated in (a), is that a thin, stiff, curved metal strip with low thermal expansion that is coated on the outside with a thick, compliant strip of material with high thermal expansion will, when heated, tighten its curvature and thereby reduce its length. By combining these elements as in (b), one obtains a porous structure with a negative thermal expansion coefficient. After Lakes (1996).

Composites can also exhibit product properties as defined by Albers (1973). A two-phase composite material exhibits a product property if the output from one phase acts as the input for the other phase. For example, following Albers (1973); Harshé, Dougherty and Newnham (1993a, 1993b); Avellaneda and Harshé (1994); and Nan (1994), consider a composite of the magnetostrictive material CoFe$_2$O$_4$ and the piezoelectric material Barium Titanate, BaTiO$_3$. An applied magnetic field generates a strain in the CoFe$_2$O$_4$ phase, which in turn generates a strain in the Barium Titanate phase, which thus produces an electric field. Thus the composite as a whole exhibits a magnetoelectric effect, where an applied magnetic effect generates an electric field, although neither phase individually exhibits such an effect. As another example, consider a composite of a phase with a large thermal expansion coefficient combined with the piezoelectric material Barium Titanate. A temperature increase generates a strain in the first phase, which in turn causes the Barium Titanate to generate an electric field. Thus the composite as a whole exhibits a pyroelectric effect, where a temperature increase generates an electric field.

Sometimes it is advantageous to have composites with structure on a hierarchy of length scales. One sees such structural hierarchy in man-made structures such as the Eiffel tower and in biological structures such as bones and tendons (Lakes 1993). In this book we will come across many examples of optimal composites that have structural hierarchy. We will often explore the limits of what is possible by considering composites with structure on infinitely many length scales. These could be approximated by more realistic composites with structure on finitely many length scales.

It is hard to look into the future, but undoubtedly it will become increasingly possible to produce "designer composites," where the microstructure has been tailored to produce desirable properties. Obviously a better understanding of the link between the microstructure and the macroscopic properties will be essential in this endeavor.

1.3. The effective tensors of composites

At present, quantum mechanics and quantum field theory provide the best description of matter on atomic, or subatomic, length scales. Yet it is well beyond the capability of modern computers to make a full simulation of the quantum mechanical equations to analyze the behavior of macroscopic bodies. The wave function for the electrons alone is described by a function in a $3N$-dimensional space, where N is the number of electrons in the body. (The factor of three arises because each electron has three spatial degrees of freedom.) The situation becomes worse when one brings the protons and neutrons into the picture, allowing for interactions with the electromagnetic, strong, and weak fields. All of this complexity is avoided when we use the equations of macroscopic physics, which can be regarded as homogenized quantum mechanical equations.

The situation is quite similar in composite materials, where, instead of using the equations of classical physics at the microscopic level, we use homogenized or effective equations at the macroscopic level. For example, in the context of electrical conductivity in a periodic microgeometry the microscopic equations, in the absence of internal current sources, take the form

$$\boldsymbol{j}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}(\boldsymbol{x}), \quad \nabla \cdot \boldsymbol{j} = 0, \quad \nabla \times \boldsymbol{e} = 0, \tag{1.1}$$

where $\boldsymbol{j}(\boldsymbol{x})$ is the current field, $\boldsymbol{e}(\boldsymbol{x}) = \nabla\phi(\boldsymbol{x})$ is the electric field, $-\phi(\boldsymbol{x})$ is the electrical potential, and $\boldsymbol{\sigma}(\boldsymbol{x})$ is the conductivity tensor field. The first equation in (1.1) is called the constitutive relation. It governs the relation between the fields $\boldsymbol{j}(\boldsymbol{x})$ and $\boldsymbol{e}(\boldsymbol{x})$, which satisfy the differential constraints imposed by the last two equations in (1.1). To avoid carrying around minus signs we will simply refer to $\phi(\boldsymbol{x})$ as the electrical potential, although it should be kept in mind that it is actually $-\phi(\boldsymbol{x})$, which is the electrical potential. All of these fields have rapid oscillations on the length scale of the microstructure, and possibly slow variations on a much larger length scale. At the macroscopic level the equations take the same basic form:

$$\boldsymbol{j}_0(\boldsymbol{x}) = \boldsymbol{\sigma}_*\boldsymbol{e}_0(\boldsymbol{x}), \quad \nabla \cdot \boldsymbol{j}_0 = 0, \quad \nabla \times \boldsymbol{e}_0 = 0, \tag{1.2}$$

where $\boldsymbol{j}_0(\boldsymbol{x})$ and $\boldsymbol{e}_0(\boldsymbol{x})$ are local averages of $\boldsymbol{j}$ and $\boldsymbol{e}$ over a cube centered at $\boldsymbol{x}$, having size large compared with the microstructure (we will make this more precise in subsequent sections). These averaged fields have the oscillations on the length scale of the microstructure smoothed out, but they retain slow variations. The first equation is the effective constitutive relation. The tensor field $\boldsymbol{\sigma}_*$ appearing in it is called the effective conductivity tensor of the medium because on a macroscopic length scale the composite behaves exactly like a homogeneous medium with conductivity $\boldsymbol{\sigma}_*$, which only has variations on the macroscopic scale. It is defined through the solution to a cell problem. One looks for pairs of *periodic fields* $\boldsymbol{j}(\boldsymbol{x})$ and $\boldsymbol{e}(\boldsymbol{x})$, which solve the conductivity equations in the periodic microgeometry. The relation

$$\langle \boldsymbol{j} \rangle = \boldsymbol{\sigma}_* \langle \boldsymbol{e} \rangle$$

between the volume averages $\langle \boldsymbol{j} \rangle$ and $\langle \boldsymbol{e} \rangle$ of each pair $\boldsymbol{j}$ and $\boldsymbol{e}$; when evaluated for sufficiently many pairs, it serves to define the effective tensor $\boldsymbol{\sigma}_*$. Thus the problem of solving (1.1) is decoupled into the problem of solving the macroscopic equations (1.2) and the problem of solving the microscopic cell problem. This decoupling makes numerical solutions much easier, and also allows one to intuitively think of the medium as effectively a homogeneous medium with conductivity $\boldsymbol{\sigma}_*$.

The effective tensor $\boldsymbol{\sigma}_*$ is not just a simple local average of $\boldsymbol{\sigma}(\boldsymbol{x})$ but instead depends on it nonlinearly. The problem of determining $\boldsymbol{\sigma}_*$ from $\boldsymbol{\sigma}(\boldsymbol{x})$ is a nontrivial problem, even when only two isotropic conducting materials are present, that is, when $\boldsymbol{\sigma}(\boldsymbol{x})$ only takes the two values $\sigma_1 \boldsymbol{I}$ and $\sigma_2 \boldsymbol{I}$, in which $\boldsymbol{I}$ is the identity tensor and σ_1 and σ_2 are both positive. One might hope that if the two constituent materials were sufficiently "well mixed" there would be a universal mean-field formula giving $\boldsymbol{\sigma}_*$ in terms of σ_1, σ_2 and the volume fractions occupied by the materials. However, this is not the case. In the mixing of materials there is nothing equivalent to the Gibbs distribution in statistical physics. The probabilities of different configurations are highly dependent on the process by which the composite is formed (of which one typically has limited knowledge). An approximation for $\boldsymbol{\sigma}_*$ that works well for one class of materials will fail for another class of materials. The main focus of this book is how the behavior of tensor fields, such as $\boldsymbol{\sigma}(\boldsymbol{x})$, on the microscopic scale influence the behavior of the associated effective tensors, such as $\boldsymbol{\sigma}_*$, on the macroscopic scale.

It may happen that the equations on a macroscopic scale take a different form than the equations on the microscopic scale. For example, in a porous medium, and for low flow rates, the Stokes equations describe the fluid flow on the microscopic level whereas Darcy's law (which says that the fluid velocity is a linear function of the pressure gradient) describes the fluid flow on the macroscopic level. We do not investigate such equations in this book. We instead focus on sets of equations that have the same form on the microscopic and macroscopic levels, consisting of fields linked by a constitutive equation, and satisfying appropriate differential constraints. In the conductivity example, the constitutive equation is the relation $\boldsymbol{j} = \boldsymbol{\sigma}\boldsymbol{e}$, and the fields $\boldsymbol{j}$ and $\boldsymbol{e}$ satisfy the differential constraints that $\nabla \cdot \boldsymbol{j} = 0$ and $\nabla \times \boldsymbol{e} = 0$.

We now endeavor to clarify the concepts of homogenization and effective tensors from various different viewpoints. For simplicity we confine our attention to the conductivity problem. The extension to the various other physical equations described in the next chapter is straightforward. The descriptions given here are sketchy, and are not meant to be a substitute for the many books on homogenization. On the other hand, a deep understanding of homogenization theory is not necessary for following the rest of the book, so don't worry if you can't understand some of the approaches outlined here.

1.4. Homogenization from an intuitive viewpoint

Homogenization from an intuitive viewpoint is described in the review article of Hashin (1983) and in the book of Nemat-Nasser and Hori (1999). We need to introduce three length scales:

- The microscale is characterized by lengths less than ℓ_1, which must be chosen greater than the maximum size of inhomogeneities in the microstructure.
- The mesoscale, which is some intermediate length scale, is characterized by some length ℓ_2, at which the composite appears "statistically homogeneous," and at which the macroscopic fields have a slow variation. It is a convenient length scale for carrying out the mathematical analysis.
- The macroscale is characterized by lengths greater than ℓ_3, which must be chosen less than the relevant dimensions of the body Σ being examined and less than the scale of variations in the macroscopic structure of the composite.

It is assumed that these three length scales are well-separated:

$$\ell_1 \ll \ell_2 \ll \ell_3.$$

Inside the composite is a potential ϕ and an associated field $\nabla\phi$ satisfying the elliptic equation

$$\nabla \cdot \boldsymbol{\sigma} \nabla \phi = \rho \ \text{ within } \Sigma, \tag{1.3}$$

and subject to, say, the Dirichlet boundary condition $\phi(\boldsymbol{x}) = \psi(\boldsymbol{x})$ at the boundary of the body Σ, where $\psi(\boldsymbol{x})$ is some prescribed potential. The length scales have been defined so that the conductivity tensor field $\boldsymbol{\sigma}(\boldsymbol{x})$ has variations on the microscale and possibly on the macroscale, but no significant variations on the mesoscale. It is assumed that the source term $\rho(x)$ and the prescribed values of $\psi(\boldsymbol{x})$ at the boundary of the body have variations only on the macroscale, that is, only on length scales greater than ℓ_3.

On the mesoscale we introduce the smoothing operation of local averaging. Let $\Omega(\boldsymbol{x})$ denote a mesosized cubic window of side ℓ_2 and volume $|\Omega| = \ell_2^3$ centered at the point $\boldsymbol{x}$. Given a field $\boldsymbol{P}$, we define

$$\langle \boldsymbol{P} \rangle_\Omega(\boldsymbol{x}) = \frac{1}{|\Omega|} \int_{\Omega(\boldsymbol{x})} \boldsymbol{P}(\boldsymbol{x}')d\boldsymbol{x}' = \frac{1}{|\Omega|} \int_{\Omega(0)} \boldsymbol{P}(\boldsymbol{x}+\boldsymbol{y})d\boldsymbol{y} \tag{1.4}$$

as the average of $\boldsymbol{P}$ over the window, where $\boldsymbol{y} = \boldsymbol{x}' - \boldsymbol{x}$. Suppose, for example, that we have a curl free field $\boldsymbol{e}(\boldsymbol{x})$. Since the locally averaged field $\langle \boldsymbol{e} \rangle_\Omega(\boldsymbol{x})$ is an average over a set of fields $\boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y})$ parameterized by $\boldsymbol{y}$, each of which is a translation of $\boldsymbol{e}(\boldsymbol{x})$ and therefore curl free, it follows that $\langle \boldsymbol{e} \rangle_\Omega(\boldsymbol{x})$ is also curl free. Thus local averaging preserves the differential constraints on the fields. The locally averaged field is defined by (1.4) only within a region Σ' inside Σ consisting of all points $\boldsymbol{x}$ such that $\Omega(\boldsymbol{x})$ lies entirely inside Σ. Since $\ell_2 \ll \ell_3$, the boundary of Σ' will be close to the boundary of Σ and the locally averaged fields will have some smooth extension to the boundary of Σ (we are not trying to be too precise here).

The basic idea behind homogenization is that when the three length scales are well-separated the elliptic equation decouples into an equation on the macroscopic scale and a set of equations on the mesoscale. The expectation is that on the macroscopic scale the local average $\langle \phi \rangle_\Omega$ of the potential ϕ satisfies

$$\nabla \cdot \boldsymbol{\sigma}_* \nabla \langle \phi \rangle_\Omega = \rho \ \text{ within } \Sigma, \quad \langle \phi \rangle_\Omega = \psi \ \text{ on } \partial\Sigma,$$

for some appropriate choice of the effective conductivity tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$, which only has variations on the macroscale, and which only depends on values of $\boldsymbol{\sigma}$ within the window $\Omega(\boldsymbol{x})$, and not on ρ or ψ. Now, since the effective tensor $\boldsymbol{\sigma}_*(\boldsymbol{x})$ only has variations on the macroscopic scale, we only need calculate it at a set of representative sample points (avoiding those points where $\Omega(\boldsymbol{x})$ intersects the boundary of Σ) and then smoothly interpolate the function between these points and to the boundary of Σ. Then, because $\boldsymbol{\sigma}_*(\boldsymbol{x})$ only depends on $\boldsymbol{\sigma}$ within the window, we can calculate it at each representative sample point by considering a periodic medium obtained by periodically extending the material within the box $\Omega(\boldsymbol{x})$ and looking for solutions of the conductivity equation (1.3) with $\rho = 0$ and $\nabla\phi$ being periodic. The effective tensor is obtained through the relation between the average fields,

$$\langle \boldsymbol{\sigma}\nabla\phi\rangle_\Omega = \boldsymbol{\sigma}_*\langle\nabla\phi\rangle_\Omega, \tag{1.5}$$

within this periodic medium. This relation defines the effective tensor at each representative sample point.

The decoupling of the equations means that we can replace the composite that has rapid oscillations in its moduli by an effective material with a slowly varying effective tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$ without changing the local averages of the fields. The idea is that this approximation should be good when the scales are well-separated, and exact in the limit as ℓ_2/ℓ_1 and ℓ_3/ℓ_2 approach infinity.

Incidentally, notice that when $\rho = 0$ any solution to the conductivity equations (1.3) remains a solution when $\boldsymbol{\sigma}(\boldsymbol{x})$ is replaced by $\boldsymbol{\sigma}'(\boldsymbol{x}) = \lambda\boldsymbol{\sigma}(\boldsymbol{x})$. It then follows from (1.5) that this medium with conductivity $\boldsymbol{\sigma}'$ will have effective conductivity $\boldsymbol{\sigma}'_* = \lambda\boldsymbol{\sigma}_*$. In other words, the effective conductivity has the homogeneity property that

$$\boldsymbol{\sigma}'_* = \lambda\boldsymbol{\sigma}_* \text{ when } \boldsymbol{\sigma}' = \lambda\boldsymbol{\sigma}. \tag{1.6}$$

1.5. Periodic homogenization

The intuitive viewpoint, while making good physical sense, needs some mathematical justification. A partial justification of the intuitive viewpoint is provided in the context of periodic homogenization. Periodic homogenization is described in the books of Bensoussan, Lions, and Papanicolaou (1978); Sanchez-Palencia (1980); Bakhvalov and Panasenko (1989); Persson, Persson, Svanstedt, and Wyller (1993); and Zhikov, Kozlov, and Oleinik (1994). The two-scale and multiscale treatments of Nguetseng (1989), Allaire (1992), and Allaire and Briane (1996), provide a rigorous basis for the method. One considers a sequence of problems, parameterized by a variable ϵ that, roughly speaking, corresponds to the size of the microstructure, and one examines what happens in the limit as ϵ tends to zero. The conductivity tensor field and the potential are assumed to be functions of both the variable $\boldsymbol{x}$, called the macroscopic or slow variable, and the variable $\boldsymbol{y} = \boldsymbol{x}/\epsilon$, called the microscopic or fast variable. Roughly speaking, the dependence of the fields on $\boldsymbol{x}$ captures their macroscopic variation while the dependence on $\boldsymbol{y}$ captures their microscopic or local variation.

In each problem the body Σ is filled by a material with conductivity tensor

$$\boldsymbol{\sigma}_\epsilon(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{x}/\epsilon), \tag{1.7}$$

where for fixed $\boldsymbol{x}$, $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})$ is periodic in the variable $\boldsymbol{y}$, say, with a square unit cell of side length h independent of $\boldsymbol{x}$. When ϵ is very small this means that $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ is almost periodic in $\boldsymbol{x}$ on the microscale, that is, on length scales of the order of h/ϵ. Some insight into the geometrical interpretation of this can be gained by considering the case when $\boldsymbol{x}$ and $\boldsymbol{y}$ are one-dimensional variables. Then, as sketched in figure 1.3, $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})$ is defined on a cylinder and $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ represents the values of $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})$ along a tightly wound spiral on the cylinder, which gets tighter as $\epsilon \to 0$. (This geometrical interpretation was communicated to me by Luc Tartar.)

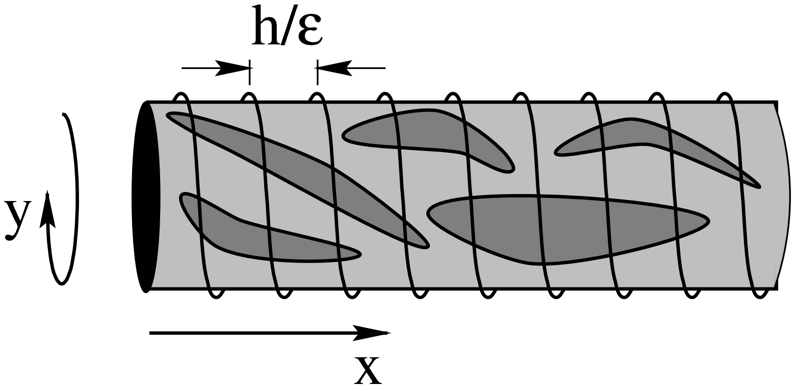


Figure 1.3. When $\boldsymbol{x}$ and $\boldsymbol{y}$ are one-dimensional, the function $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})$ can be regarded as lying on the surface of a cylinder of circumference h. As illustrated here, the function $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})$ could, for example, take two different values, one value in the darker shaded region and a different value in the lighter shaded region. The function $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{x}/\epsilon)$ represents the values of the function along a tightly wound spiral on the cylinder.

To obtain the right homogenized equations one has to be careful. This is illustrated by the following example. Consider a connected cubic network of thin conducting rods, which is diagonally displaced from a second connected cubic network of conducting rods, with a different effective tensor, so that the two networks do not touch each other. The surrounding material is assumed to be nonconducting. It is clear that the electrical potential field on the second network is independent of the potential field on the first network, and thus one would expect the homogenized equations to be a pair of uncoupled conductivity equations, one for each network. If there are suitably thin bridges linking the two networks, then Khruslov (1978), Briane (1998), and Briane and Mazliak (1998) have shown that the homogenized equations are coupled.

Some restrictions of the conductivity field $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ are needed to avoid such strange homogenized equations. For simplicity we assume that $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ is a symmetric matrix-valued function satisfying the ellipticity condition

$$\alpha \boldsymbol{I} \leq \boldsymbol{\sigma}_\epsilon(\boldsymbol{x}) \leq \beta \boldsymbol{I} \quad \text{for all } \epsilon \text{ and } \boldsymbol{x}, \tag{1.8}$$

for some positive α and β independent of both ϵ and $\boldsymbol{x}$. We then have a sequence of electrical potentials $\phi_\epsilon(\boldsymbol{x})$ satisfying the elliptic equations

$$\nabla \cdot \boldsymbol{\sigma}_\epsilon \nabla \phi_\epsilon = \rho \ \text{ within } \Sigma, \quad \phi_\epsilon = \psi \ \text{ on } \partial\Sigma, \tag{1.9}$$

where the source term $\rho(\boldsymbol{x})$ and the potential $\psi(\boldsymbol{x})$ at the boundary of Σ are prescribed and assumed to be independent of ϵ.

To solve for the potential in the limit $\epsilon \to 0$ one uses a multiple-scale analysis and looks for a solution of the form

$$\phi_\epsilon(\boldsymbol{x}) = \phi_0(\boldsymbol{x}, \boldsymbol{x}/\epsilon) + \epsilon\phi_1(\boldsymbol{x}, \boldsymbol{x}/\epsilon) + \epsilon^2\phi_2(\boldsymbol{x}, \boldsymbol{x}/\epsilon) + \cdots,$$

where for fixed $\boldsymbol{x}$ the functions $\phi_i(\boldsymbol{x}, \boldsymbol{y})$, $i = 0, 1, 2, \ldots$, are periodic functions of $\boldsymbol{y}$ with the same periodicity as $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})$. By substituting this expansion and (1.7) into (1.9) and separating terms having coefficients sharing the same power of ϵ one obtains a series of equations. I will not go through the analysis, since it is contained in the above-cited books. One finds that $\phi_0(\boldsymbol{x}, \boldsymbol{y})$ depends only on $\boldsymbol{x}$, that is, $\phi_0(\boldsymbol{x}, \boldsymbol{y}) = \phi_0(\boldsymbol{x})$, where $\phi_0(\boldsymbol{x})$ satisfies

$$\nabla \cdot \boldsymbol{\sigma}_* \nabla\phi_0 = \rho \text{ within } \Sigma, \quad \phi_0 = \psi \text{ on } \partial\Sigma,$$

in which the effective conductivity tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$ is obtained at each point $\boldsymbol{x}$ by solving the auxiliary "cell problem." Given an applied field $\boldsymbol{e}_0$, the cell problem consists of finding the function $\phi_1(\boldsymbol{e}_0; \boldsymbol{y})$ periodic in $\boldsymbol{y}$, which solves

$$\nabla_y \cdot \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})\nabla_y[\boldsymbol{e}_0 \cdot \boldsymbol{y} + \phi_1(\boldsymbol{e}_0; \boldsymbol{y})] = 0,$$

where $\nabla_y\cdot$ and ∇_y are the divergence and gradient with respect to $\boldsymbol{y}$, keeping $\boldsymbol{x}$ fixed. Once the cell problem is solved for a basis of applied fields $\boldsymbol{e}_0$, the effective tensor is obtained through the relation

$$\langle \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})\nabla_y[\boldsymbol{e}_0 \cdot \boldsymbol{y} + \phi_1(\boldsymbol{e}_0; \boldsymbol{y})]\rangle = \boldsymbol{\sigma}_*\langle \nabla_y[\boldsymbol{e}_0 \cdot \boldsymbol{y} + \phi_1(\boldsymbol{e}_0; \boldsymbol{y})]\rangle,$$

where the averages are over $\boldsymbol{y}$, keeping $\boldsymbol{x}$ fixed. Since ϕ_1 is periodic in $\boldsymbol{y}$, it follows that the average value of $\nabla_y\phi_1$ is zero. Therefore we can rewrite the relation as

$$\langle \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y})\nabla_y[\boldsymbol{e}_0 \cdot \boldsymbol{y} + \phi_1(\boldsymbol{e}_0; \boldsymbol{y})]\rangle = \boldsymbol{\sigma}_*\boldsymbol{e}_0.$$

The next function appearing in the series expansion is found to be

$$\phi_1(\boldsymbol{x}, \boldsymbol{y}) = \phi_1(\nabla\phi_0(\boldsymbol{x}); \boldsymbol{y}).$$

Thus the solution of the cell problem not only gives the effective tensor $\boldsymbol{\sigma}_*$ needed to compute ϕ_0, but it also gives the leading correction ϕ_1 to the potential in the series expansion. The results of this multiple-scale analysis can be verified by more rigorous methods [see, for example, Bensoussan, Lions, and Papanicolaou (1978); Tartar (1978); Nguetseng (1989); Allaire (1992); Allaire and Briane (1996); and Murat and Tartar (1997)]. One can extend the analysis to higher powers in ϵ and thereby obtain a solution that captures the correct asymptotic behavior as $\epsilon \to 0$. A rigorous justification of the resulting higher order homogenized solutions has been provided by Bakhvalov and Panasenko (1989). Smyshlyaev and Cherednichenko (2000) show how the higher order homogenized solutions also can be justified variationally, by using an energy that depends on higher order gradients.

Notice that while the potential $\phi_\epsilon(\boldsymbol{x})$ converges to $\phi_0(\boldsymbol{x})$ as $\epsilon \to 0$, the electric field $\nabla\phi_\epsilon(\boldsymbol{x})$ does not converge to $\nabla\phi_0(\boldsymbol{x})$. Indeed, $\nabla\phi_\epsilon(\boldsymbol{x})$ has rapid oscillations on the microscale, whereas $\nabla\phi_0(\boldsymbol{x})$ does not. Nevertheless, there is convergence in a weak sense. A sequence of fields $\boldsymbol{e}_\epsilon(\boldsymbol{x})$ is said to weakly converge to $\boldsymbol{e}_0(\boldsymbol{x})$ if

$$\lim_{\epsilon \to 0} \int g(\boldsymbol{x})\boldsymbol{e}_\epsilon(\boldsymbol{x})d\boldsymbol{x} = \int g(\boldsymbol{x})\boldsymbol{e}_0(\boldsymbol{x})d\boldsymbol{x},$$

for all square integrable test functions $g(\boldsymbol{x})$. With this definition $\nabla\phi_\epsilon(\boldsymbol{x})$ converges weakly to $\nabla\phi_0(\boldsymbol{x})$. Roughly speaking, taking the weak limit of a sequence of functions smoothes out the rapid oscillations; the weak limit represents the locally averaged function. To clarify the concept of weak convergence, let us consider a simple mathematical example, which is not intended to have any deeper physical significance: For any vector $\boldsymbol{n} \neq 0$ the sequence of functions

$$f_\epsilon(\boldsymbol{x}) = \frac{2\cos^2(\boldsymbol{x}\cdot\boldsymbol{n}/\epsilon)}{|\boldsymbol{x}|^4+1}$$

converges weakly as $\epsilon \to 0$ to the function

$$f_0(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|^4+1}.$$

Of course the assumption (1.8) is too strong since it excludes all composite materials with voids. The homogenization of such perforated structures, including cellular materials, is treated in the books of Oleinik, Shamaev, and Yosifian (1992); Zhikov, Kozlov, and Oleinik (1994); and Cioranescu and Saint Jean Paulin (1999).

1.6. Homogenization in random media

In random media the analog of periodicity is statistical homogeneity or stationarity. Homogenization in random media is described with varying degrees of rigor in the books of Beran (1968); Bensoussan, Lions, and Papanicolaou (1978); Bakhvalov and Panasenko (1989); and Zhikov, Kozlov, and Oleinik (1994), and in the papers of Kozlov (1978), Papanicolaou and Varadhan (1982), and Golden and Papanicolaou (1983).

The conductivity tensor field $\boldsymbol{\sigma}(\boldsymbol{x},\omega)$ is a function of both $\boldsymbol{x}$ and the particular microstructure realization ω in the ensemble being considered. It is stationary if, given any set of points $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_m$ and any vector $\boldsymbol{h}$, the joint distribution of the set of tensors

$$\boldsymbol{\sigma}(\boldsymbol{x}_1,\omega), \boldsymbol{\sigma}(\boldsymbol{x}_2,\omega), \dots, \boldsymbol{\sigma}(\boldsymbol{x}_m,\omega)$$

and the joint distribution of the set of tensors

$$\boldsymbol{\sigma}(\boldsymbol{x}_1+\boldsymbol{h},\omega), \boldsymbol{\sigma}(\boldsymbol{x}_2+\boldsymbol{h},\omega), \dots, \boldsymbol{\sigma}(\boldsymbol{x}_m+\boldsymbol{h},\omega)$$

are the same as ω varies over all realizations weighted according to a probability measure $\mathcal{P}(\omega)$. This stationarity will be guaranteed if $\mathcal{P}(\omega)$ is invariant under translations of the microstructure, that is, if, roughly speaking, a given microstructure and the translated microstructure have the same probability of occurring. The case of periodic homogenization, where $\boldsymbol{\sigma}(\boldsymbol{x})$ is periodic in $\boldsymbol{x}$, can be treated in this framework by letting the ensemble consist of $\boldsymbol{\sigma}(\boldsymbol{x})$ and all of its translates $\boldsymbol{\sigma}(\boldsymbol{x}+\boldsymbol{h})$ weighted with uniform probability density $\mathcal{P}$.

The equations of interest are now

$$\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{x}/\epsilon,\omega)\nabla\phi_\epsilon(\boldsymbol{x},\omega) = \rho(\boldsymbol{x}), \quad \phi_\epsilon(\boldsymbol{x},\omega) = \psi(\boldsymbol{x}) \text{ on } \partial\Sigma,$$

as ω varies over all microstructure realizations. Here ϵ can be regarded as a (continuous or discrete) parameter that sets the scale of the microstructure but does not necessarily represent a characteristic length. The homogenized fields are defined by the limits

$$\begin{aligned}
\phi_0(\boldsymbol{x}) &= \lim_{\epsilon\to 0} E\{\phi_\epsilon(\boldsymbol{x},\omega)\},\\
\boldsymbol{e}_0(\boldsymbol{x}) &= \lim_{\epsilon\to 0} E\{\nabla\phi_\epsilon(\boldsymbol{x},\omega)\},\\
\boldsymbol{j}_0(\boldsymbol{x}) &= \lim_{\epsilon\to 0} E\{\boldsymbol{\sigma}(\boldsymbol{x}/\epsilon,\omega)\nabla\phi_\epsilon(\boldsymbol{x},\omega)\},
\end{aligned}$$

where $E\{\cdot\}$ denotes the expectation value, or ensemble average:

$$E\{f(\boldsymbol{x},\omega)\} \equiv \int f(\omega)\mathcal{P}(\omega)d\omega,$$

where the integration is over all realizations ω in the ensemble. The homogenized potential $\phi_0(\boldsymbol{x})$ satisfies the homogenized equation

$$\nabla\cdot\boldsymbol{\sigma}_*\nabla\phi_0 = \rho \ \text{ within } \Sigma, \quad \phi_0(\boldsymbol{x}) = \psi(\boldsymbol{x}) \ \text{ on } \partial\Sigma,$$

or, equivalently,

$$\boldsymbol{j}_0 = \boldsymbol{\sigma}_*\boldsymbol{e}_0, \quad \nabla\cdot\boldsymbol{j}_0 = \rho, \quad \boldsymbol{e}_0 = \nabla\phi_0, \quad \phi_0(\boldsymbol{x}) = \psi(\boldsymbol{x}) \ \text{ on } \partial\Sigma,$$

where the effective tensor $\boldsymbol{\sigma}_*$ is independent of $\boldsymbol{x}$ and ω and is determined by finding *stationary* potentials $\phi_1(\boldsymbol{x},\omega)$ that solve

$$\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{x}/\epsilon,\omega)[\boldsymbol{e}_0 + \nabla\phi_1(\boldsymbol{x},\omega)] = 0,$$

for various constant fields $\boldsymbol{e}_0$ (independent of $\boldsymbol{x}$ and ω), by calculating

$$\boldsymbol{j}_0 = E\{\boldsymbol{\sigma}(\boldsymbol{x}/\epsilon,\omega)[\boldsymbol{e}_0 + \nabla\phi_\epsilon(\boldsymbol{x},\omega)]\},$$

which gives $\boldsymbol{\sigma}_*$ via the relation

$$\boldsymbol{j}_0 = \boldsymbol{\sigma}_*\boldsymbol{e}_0.$$

When ϵ is small the potential $\phi_0(\boldsymbol{x})$ is not just a good approximation to the expectation value of $\phi_\epsilon(\boldsymbol{x},\omega)$. Kozlov (1978) and Papanicolaou and Varadhan (1982) establish the stronger result that

$$\int_\Sigma E\{|\phi_\epsilon(\boldsymbol{x},\omega) - \phi_0(\boldsymbol{x})|^2\}d\boldsymbol{x} = 0,$$

which shows that when ϵ is sufficiently small $\phi_0(\boldsymbol{x})$ is almost everywhere a good approximation to nearly every field $\phi_\epsilon(\boldsymbol{x},\omega)$ in the ensemble.

Papanicolaou and Varadhan (1982) and Golden and Papanicolaou (1983) have established that this definition of the effective conductivity tensor is consistent with the more usual definition where a cubic sample of the composite is taken and then $\boldsymbol{\sigma}_*$ is obtained in an infinite volume limit as the size the cube tends to infinity.

1.7. Homogenization in the settings of G-, H-, and Γ-convergence

The sequences of microstructures associated with periodic homogenization and homogenization in random media are rather special. Each element in the sequence $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{x}/\epsilon)$ or the sequence $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x},\omega) = \boldsymbol{\sigma}(\boldsymbol{x}/\epsilon,\omega)$ has oscillations in the microstructure on lengths of the order of ϵ. One might wonder if homogenization can be generalized in some way to essentially arbitrary sequences of tensor fields. The frameworks of G-, H-, and Γ-convergence provide such a generalization. Spagnolo (1968) first introduced G-convergence. It is associated with the convergence of Green's functions, which is what the G signifies. Murat and Tartar (Tartar 1978; Murat and Tartar 1985, 1997) introduced H-convergence (H for homogenization), which permits problems with nonsymmetric conductivity tensors to be treated. De Giorgi (1975) introduced Γ-convergence, which is an abstract notion of functional convergence, not just limited to homogenization. These approaches provide a rigorous justification for the intuitive viewpoint of homogenization in a very general setting. They are described, for example, in the articles of De Giorgi (1984), Allaire (1997), and Murat and Tartar (1997); in the books of Buttazzo (1989), Dal Maso (1993), Zhikov, Kozlov, and Oleinik (1994), and Attouch (1984); and in the lecture notes of Raitums (1997). Allaire (1997), in particular, provides an excellent short summary of the different approaches.

To define G-, H-, and Γ-convergence consider a sequence of symmetric second-order tensor fields $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ such that (1.8) is satisfied for some choice of positive constants α and β. This sequence is said to G-converge to a symmetric tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$ if and only if for any $\rho(\boldsymbol{x})$ (in an appropriate space) the potentials ϕ_ϵ that solve the Dirichlet problem

$$\nabla \cdot \boldsymbol{\sigma}_\epsilon \nabla \phi_\epsilon = \rho \ \text{ within } \Sigma, \quad \phi_\epsilon = 0 \ \text{ on } \partial\Sigma,$$

converge to a potential $\nabla\phi_0$ that satisfies

$$\nabla \cdot \boldsymbol{\sigma}_* \nabla \phi_0 = \rho \ \text{ within } \Sigma, \quad \phi_0 = 0 \ \text{ on } \partial\Sigma.$$

This definition is motivated by the theorem of Spagnolo (1968) that *any* sequence $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ satisfying (1.8) has a subsequence that G-converges to a tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$. Roughly speaking, in those regions where $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ converges to a fixed tensor field, $\boldsymbol{\sigma}_*(\boldsymbol{x})$ equals that tensor field; whereas in those regions where $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ develops oscillations on finer and finer length scales, $\boldsymbol{\sigma}_*(\boldsymbol{x})$ equals the associated effective tensor. If in a subregion the value of $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ alternates between two different fixed values as ϵ is decreased, one chooses the subsequence so that only one of these fixed values is selected.

The sequence is said to H-converge to a symmetric tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$ if and only if on each bounded domain Σ all pairs of sequences of square integrable vector fields $\boldsymbol{j}_\epsilon$ and $\boldsymbol{e}_\epsilon$ on Σ for which

- $\boldsymbol{j}_\epsilon = \boldsymbol{\sigma}_\epsilon \boldsymbol{e}_\epsilon$ is satisfied for all ϵ;
- $\nabla \cdot \boldsymbol{j}_\epsilon$ and $\nabla \times \boldsymbol{e}_\epsilon$ remain within compact sets in appropriate Hilbert spaces (which guarantees that $\nabla \cdot \boldsymbol{j}_\epsilon$ and $\nabla \times \boldsymbol{e}_\epsilon$ do not oscillate too much as $\epsilon \to 0$);
- $\boldsymbol{j}_\epsilon$ and $\boldsymbol{e}_\epsilon$ converge weakly as ϵ tends to zero;

have weak limits $\boldsymbol{j}_0$ and $\boldsymbol{e}_0$ satisfying

$$\boldsymbol{j}_0 = \boldsymbol{\sigma}_* \boldsymbol{e}_0.$$

Again, *any* sequence $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ satisfying (1.8) has a subsequence that H-converges to a tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$ (Tartar 1978; Murat and Tartar 1985, 1997).

The definition of Γ-convergence is more abstract. In the current setting it is defined in terms of the quadratic form $(\nabla\phi_\epsilon) \cdot \boldsymbol{\sigma}_\epsilon \nabla\phi_\epsilon$, which physically represents the electrical power dissipation. The sequence $\boldsymbol{\sigma}_\epsilon$ is defined to Γ-converge to $\boldsymbol{\sigma}_*(\boldsymbol{x})$ if

- for any potential $\phi_0(\boldsymbol{x})$ and any sequence $\varphi_\epsilon(\boldsymbol{x})$ such that $\varphi_\epsilon(\boldsymbol{x})$ converges to $\phi_0(\boldsymbol{x})$, we have
$$\lim_{\epsilon\to 0} \int_\Sigma (\nabla\varphi_\epsilon) \cdot \boldsymbol{\sigma}_\epsilon \nabla\varphi_\epsilon \geq \int_\Sigma (\nabla\phi_0) \cdot \boldsymbol{\sigma}_* \nabla\phi_0;$$
- for any potential $\phi_0(\boldsymbol{x})$ there exists a sequence $\phi_\epsilon(\boldsymbol{x})$ that converges to $\phi_0(\boldsymbol{x})$ such that
$$\lim_{\epsilon\to 0} \int_\Sigma (\nabla\phi_\epsilon) \cdot \boldsymbol{\sigma}_\epsilon \nabla\phi_\epsilon = \int_\Sigma (\nabla\phi_0) \cdot \boldsymbol{\sigma}_* \nabla\phi_0.$$

Here $\varphi_\epsilon(\boldsymbol{x})$ should be regarded as a sequence of trial potentials, and the first inequality arises from variational principles for the effective tensor (see section 13.1 on page 271). The sequence φ_ϵ (called a Γ-realizing sequence) can be taken to be the solution of

$$\nabla \cdot \boldsymbol{\sigma}_\epsilon \nabla \phi_\epsilon = \nabla \cdot \boldsymbol{\sigma}_* \nabla \phi_0 \text{ within } \Sigma, \quad \phi_\epsilon = \phi_0 \text{ on } \partial \Sigma.$$

Once again, *any* sequence $\boldsymbol{\sigma}_\epsilon(\boldsymbol{x})$ satisfying (1.8) has a subsequence that Γ-converges to a tensor field $\boldsymbol{\sigma}_*(\boldsymbol{x})$ (Dal Maso 1993). One advantage of Γ-convergence is that it is not restricted to linear equations. Using it, Braides (1985) and Müller (1987) have found that the homogenized energy for nonlinear elasticity in a periodic material has to be computed from solutions in increasingly large blocks of unit cells, not just from the solutions in a single unit cell of periodicity, as in the linear case.

Various properties of G-, H-, and Γ-convergence have been established. For example, the field $\boldsymbol{\sigma}_*(\boldsymbol{x})$ inside a window Ω within Σ is unaffected by the values that the sequence $\boldsymbol{\sigma}_\epsilon$ takes outside the window. Also, $\boldsymbol{\sigma}_*(\boldsymbol{x})$ is independent of the source term $\rho(\boldsymbol{x})$.

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