Fast analytical techniques for electrical and electronic circuits

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK,
40 West 20th Street, New York, NY 10011-4211, USA
477 Wiliamstown Road, Port Melbourne VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10.5/14pt *System* Poltype® [VN]

A catalogue record for this book is available from the British Library

ISBN 0 521 62442 8 hardback

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1 Introduction

The joys of network analysis

1.1 Fast analytical methods

The universally adopted method of teaching network theory is the formal and systematic method of nodal or loop analysis. Although the matrix algebra of formal network analysis is ideal for obtaining *numerical* answers by a computer, it fails hopelessly for obtaining *analytical* answers which provide physical insight into the operation of the circuit. It is not hard to see that, when numerical values of circuit components are not given, inverting a 3×3, or higher-order, matrix with symbolic entries can be very time consuming. This is only part of the problem of matrix analysis because even if one were to survive the algebra of inverting a matrix symbolically, the answer could be an unintelligible and lengthy symbolic expression. It is important to realize that an analytical answer is not merely a symbolic expression, but an expression in which various circuit elements are grouped together in one or more of the following ways:

(a) series and parallel combinations of resistances

Example: $R_1 + R_2 \parallel (R_3 + R_4)$

(b) ratios of resistances, time constants and gains

Example: $1 + \dfrac{R}{R_3 \parallel R_4}$, $1 + \dfrac{g_m R_L}{A_o}$, $A_m\left(1 + \dfrac{\tau_1}{\tau_2}\right)$

(c) polynomials in the frequency variable, s, with a unity leading term and coefficients in terms of sums and products of time constants

Example: $1 + s(\tau_1 + \tau_2) + s^2\tau_1\tau_3$

Such analytical expressions have been called low-entropy expressions by R. D. Middlebrook[1] because they reveal useful and *recognizable* information (low noise or entropy) about the performance of the circuit. Another extremely important advantage of low-entropy expressions is that they can be easily approximated into simpler expressions which are useful for design purposes. For instance, a series-parallel combination of resistances, as in (a), can be simplified by ignoring the smaller of two resistances in a series combination and the larger of two resistances in a parallel combination. When ratios are used as in (b), they can be simplified depending on their relative magnitude to unity. Depending on the relative magnitude of time constants, frequency response characteristics as in (c) can be simplified and either factored into two real roots, with simple analytical expressions, or remain as a complex quadratic factor.

In light of the above, the aim of fast analytical techniques can be stated as follows: fast derivation of low-entropy analytical expressions for electrical circuits. The following examples illustrate the power of this new approach to circuit analysis.

1.2 Input impedance of a bridge circuit

We will determine the input resistance, R_{in}, of the bridge circuit[2] in Fig.1.1 in a few simple steps using the extra element theorem (EET). The EET[3] and its extension, the N-extra element theorem[4] (NEET), are the main basic tools of fast network analysis discussed in this book. Both of these theorems will be introduced, derived and stated in their general form in later chapters, but since the EET for an impedance function is so trivial, we will use it now to obtain an early glimpse of what lies ahead.

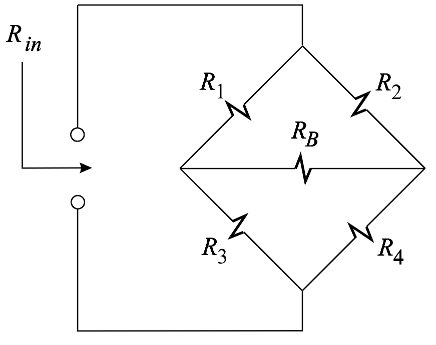

Figure 1.1

We see in Fig. 1.1 that if any one of the resistors of the bridge is zero or infinite, we can write R_{in} immediately by inspection. For instance, if we designate R_B as the extra element and let $R_B \to \infty$, as shown in Fig. 1.2*a*, we can immediately write:

$$R_{in}|_{R_B\to\infty} = (R_1 + R_3) \| (R_2 + R_4) \tag{1.1}$$

The EET now requires us to perform two additional calculations as shown in Figs. 1.2*b* and *c*. We denote the port across which the extra element is connected by (B).

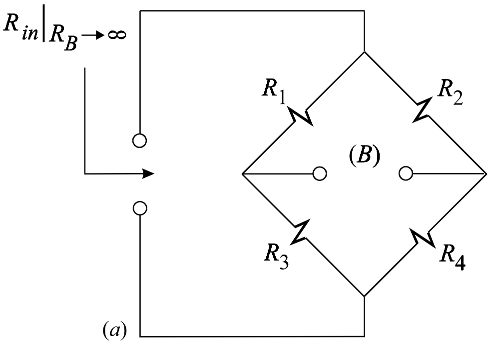

Figure 1.2

In Fig. 1.2*b*, we determine the resistance looking into the network from port (*B*) with the *input port short* and obtain by inspection:

$$\mathscr{R}^{(B)} = R_1 \parallel R_3 + R_2 \parallel R_4 \tag{1.2}$$

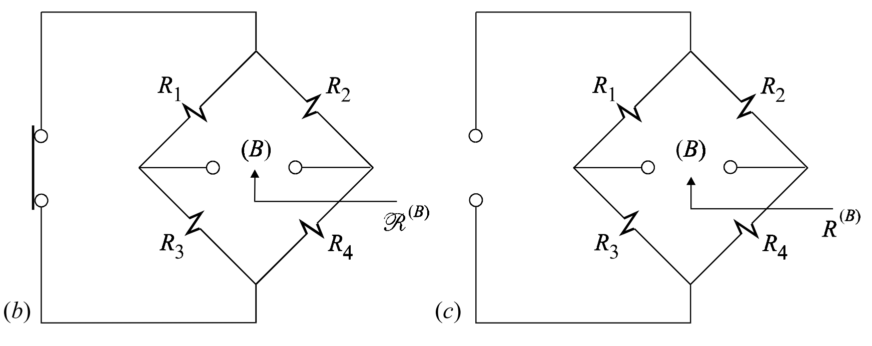

Figure 1.2 (*cont.*)

In Fig. 1.2*c*, we determine the resistance looking into the network from port (*B*) with the *input port open* and obtain by inspection:

$$R^{(B)} = (R_1 + R_2) \parallel (R_3 + R_4) \tag{1.3}$$

We now assemble these three separate and independent calculations to obtain the input resistance R_{in} in Fig. 1.1 using the following formula given by the EET:

$$R_{in} = R_{in}|_{R_B\to\infty} \frac{1 + \dfrac{\mathscr{R}^{(B)}}{R_B}}{1 + \dfrac{R^{(B)}}{R_B}} \tag{1.4}$$

Upon substituting Eqs. (1.1), (1.2) and (1.3) in (1.4):

$$R_{in} = (R_1 + R_3) \parallel (R_2 + R_4) \frac{1 + \dfrac{R_1 \parallel R_3 + R_2 \parallel R_4}{R_B}}{1 + \dfrac{(R_1 + R_2) \parallel (R_3 + R_4)}{R_B}} \tag{1.5}$$

Equation (1.5) is a low-entropy result because in it R_{in} is expressed in terms of series and parallel combinations of resistances and ratios of such resistances added to unity. Such an expression, for a given set of typical element values, can be easily approximated using rules of series and parallel combinations wherever applicable. In this expression, we can also see the contribution of the bridge resistance, R_B, to the input resistance, R_{in}, directly.

We can also appreciate two important advantages of the method of EET used in deriving R_{in} above. First, since the method of EET requires far less algebra than nodal analysis, it is considerably faster and simpler. Second, since the EET requires three separate and *independent* calculations, any *error in the analysis does not spread* and remains confined to a portion of the final answer. In a sense, this kind of analysis yields modular answers – if there is anything wrong with a particular module, it can be replaced without affecting the entire answer. This not only makes the analysis faster, but also the debugging of the analysis faster as well.

1.3 Input impedance of a bridge circuit with a dependent source

In this section we consider the effect of a dependent current source,[2,5] $g_m v_1$, in Fig. 1.3, on the input resistance R_{in}. This circuit is borrowed from a well-known

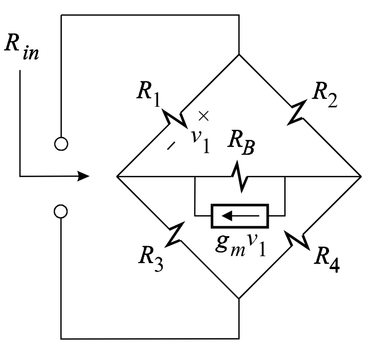

Figure 1.3

textbook by L. O. Chua and Pen-Min Lin[5] in which the authors determine the contribution of the transconductance, g_m, to the input resistance, R_{in}, using the parameter-extraction method. Because of the considerable amount of matrix algebra required by the parameter-extraction method, which would become prohibitively complex if all elements were in symbolic form, Chua and Lin have assigned numerical values ($R_1 = 1\,\Omega$, $R_2 = 0.2\,\Omega$, $R_3 = 0.5\,\Omega$, $R_4 = 10\,\Omega$ and $R_B = 0.1\,\Omega$) to all the resistors and determined:

$$R_{in} = \frac{96.3 + 5.1g_m}{137.7 + 10.5g_m}\,\Omega \tag{1.6}$$

We will now show how to determine R_{in} in three simple steps by applying the EET to the dependent current source $g_m v_1$. To demonstrate the superior power of this method of analysis, we will keep all circuit elements in symbolic form.

In Fig. 1.3, we designate the dependent current source as the extra element and set it to zero by letting $g_m = 0$. This reduces the circuit to the bridge circuit in Section 1.2, as shown in Fig. 1.4*a*. Hence, we have from Eq. (1.5):

$$R_{in}|_{g_m \to 0} = (R_1 + R_3) \| (R_2 + R_4) \frac{1 + \dfrac{R_1 \| R_3 + R_2 \| R_4}{R_B}}{1 + \dfrac{(R_1 + R_2) \| (R_3 + R_4)}{R_B}} \tag{1.7}$$

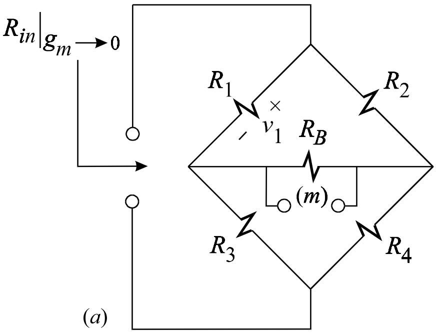

Figure 1.4

The EET now requires us to perform two additional calculations as shown in Figs. 1.4*b* and *c* in which the dependent current source is replaced with an independent one, i_m, pointing in the opposite direction. In Fig. 1.4*b* we determine the transresistance, v_1/i_m, which is the *inverse* of the transconductance gain g_m of the dependent source, with the input port short. Inspecting Fig. 1.4*b*, we see that $R_1 \| R_3$ and $R_2 \| R_4$ form a voltage divider connected across an equivalent Thevinin voltage source, $i_m R_B$, in series with a Thevinin resistance, R_B, so that we have:

$$\frac{v_1}{i_m R_B} = \frac{R_1 \parallel R_3}{R_B + R_2 \parallel R_4 + R_1 \parallel R_3} \tag{1.8}$$

It follows that the inverse gain, with the input port short, is given by:

$$\mathscr{G}^{(m)} = \left.\frac{v_1}{i_m}\right|_{(in)\to short} = \frac{R_1 \parallel R_3}{R_B + R_2 \parallel R_4 + R_1 \parallel R_3} R_B \tag{1.9}$$

Similarly, we can determine in Fig. 1.4c that the inverse gain, with the input port open, is given by:

$$\overline{G}^{(m)} = \left.\frac{v_1}{i_m}\right|_{(in)\to open} = \frac{R_B \parallel (R_3 + R_4)}{R_1 + R_2 + R_B \parallel (R_3 + R_4)} R_1 \tag{1.10}$$

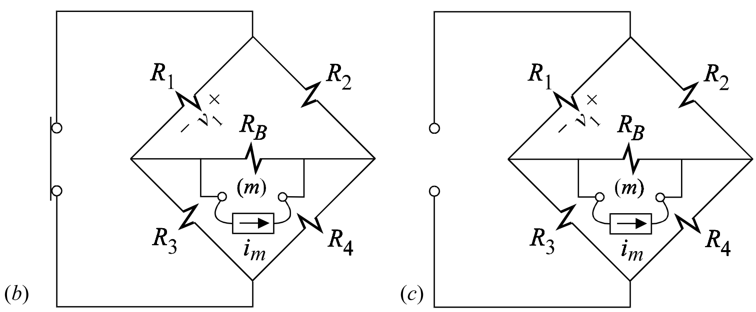

Figure 1.4 (*cont.*)

We can now assemble the final answer using the three separate calculations in Eqs. (1.7), (1.9) and (1.10) according to the following formula given by the EET:

$$R_{in} = R_{in}|_{g_m \to 0} \frac{1 + g_m \mathscr{G}^{(m)}}{1 + g_m \overline{G}^{(m)}} \tag{1.11}$$

Upon substituting, we get:

$$R_{in} = (R_1 + R_3) \parallel (R_2 + R_4) \frac{1 + \dfrac{R_1 \parallel R_3 + R_2 \parallel R_4}{R_B}}{1 + \dfrac{(R_1 + R_2) \parallel (R_3 + R_4)}{R_B}} \tag{1.12}$$

$$\times \frac{1 + \dfrac{g_m R_B}{1 + (R_B + R_2 \parallel R_4)/R_1 \parallel R_3}}{1 + \dfrac{g_m R_1}{1 + (R_1 + R_2)/R_B \parallel (R_3 + R_4)}}$$

Hence, by doing far less algebra than that required by the parameter-extraction method, we have obtained a low-entropy symbolic expression which is far superior to the one given in Eq. (1.6)

The EET, quite naturally, also allows for the value of a dependent source to become infinite so that a particular transfer becomes simplified in the same manner as that of an ideal operational amplifier circuit. In the case of R_{in} in Fig. 1.3, the EET allows us to write:

$$R_{in} = R_{in}|_{g_m \to \infty} \frac{1 + \dfrac{1}{g_m \bar{\mathscr{G}}^{(m)}}}{1 + \dfrac{1}{g_m \bar{G}^{(m)}}} \tag{1.13}$$

in which $\bar{G}^{(m)}$ and $\bar{\mathscr{G}}^{(m)}$ are the same as before and $R_{in}|_{g_m \to \infty}$ is determined in Fig. 1.5. The gain from v_1 to $g_m v_1$ reminds us of an opamp connected in some kind of

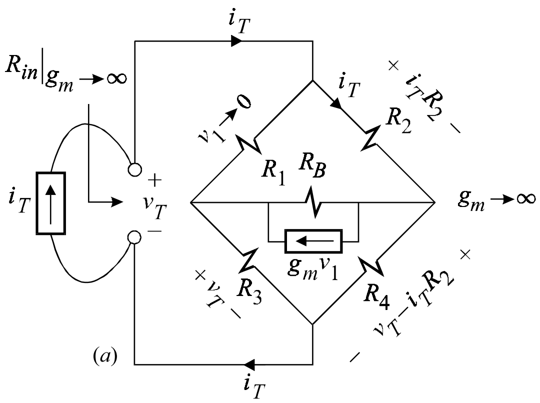

Figure 1.5

feedback fashion whose details we do not need to know at all. Now, if we let g_m become infinite, then $v_1 \to 0$ very much in the same manner as the differential input voltage of an opamp tends to zero when the gain becomes infinite and the output voltage stays finite. We can see in Fig. 1.5 that, with $g_m \to \infty$ and $v_1 \to 0$, the current through R_1becomes zero and i_T flows entirely through R_2 creating a voltage drop $i_T R_2$ across it. At the same time, v_T appears across R_3 causing a current v_T/R_3 to flow through it. We can also see that the voltage drop across R_4, when $v_1 = 0$, is equal to $v_T - i_T R_2$ so that the current through it is simply $(v_T - i_T R_2)/R_4$. Summing the currents at the lower node of the bridge, we obtain:

$$i_T = \frac{v_T}{R_3} + \frac{v_T - i_T R_2}{R_4} \tag{1.14}$$

It follows from Eq. (1.14) that:

$$\frac{v_T}{i_T} = R_{in}\big|_{g_m \to \infty} = \frac{R_3 \parallel R_4}{1 + \dfrac{R_2}{R_4}} \tag{1.15}$$

Substituting Eq. (1.15) in (1.13) we obtain another expression for R_{in} given by:

$$R_{in} = \frac{R_3 \parallel R_4}{1 + \dfrac{R_2}{R_4}} \frac{1 + \dfrac{1 + R_2 \parallel R_4/R_B}{g_m(R_B + R_2 \parallel R_4) \parallel R_1 \parallel R_3}}{1 + \dfrac{1 + R_2/R_1}{g_m(R_1 + R_2) \parallel R_B \parallel (R_3 + R_4)}} \tag{1.16}$$

Although Eq. (1.16) looks simpler than Eq. (1.12), both are very useful analytical expressions. For very small values of g_m, Eq. (1.12) is a better expression because the bilinear factor containing g_m is close to unity and R_{in} is mostly dictated by the bridge circuit. If on the other hand g_m is very large, Eq. (1.16) is a better expression because R_{in} is mostly given by Eq. (1.15), and the bilinear function of g_m in Eq. (1.16) is close to unity.

1.4 Input impedance of a reactive bridge circuit with a dependent source

Consider now the reactive bridge circuit in Fig. 1.6 for which the input impedance[2] is to be determined. By designating the capacitor as the extra element, we will show how easily $Z_{in}(s)$ can be determined by simply analyzing a few purely resistive

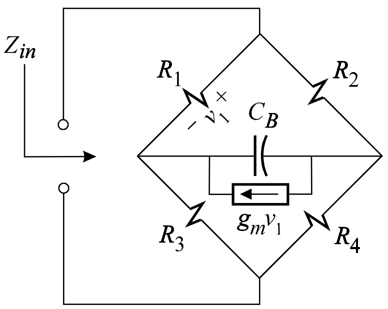

Figure 1.6

circuits. In other words, we will see how the EET allows one to determine a reactive transfer function, such as $Z_{in}(s)$, without ever having to deal with a reactive component such as $1/sC_B$. In fact, as we will see later, the most natural application of the EET and NEET is in the reduction of a circuit with N reactive elements to a set of purely resistive circuits.

If we designate $Z_B = 1/sC_B$ as the extra element and let $Z_B \to \infty$, we obtain the circuit in Fig. 1.7*a*, which is a special case of the circuit in Fig. 1.3 whose input impedance is given by Eq. (1.12). The derivation of the input impedance of the circuits in Figs. 1.3 and 1.7*a* are identical, with the exception that $R_B \to \infty$ in Fig. 1.7*a*. Hence, by letting $R_B \to \infty$ in Eq. (1.12) we obtain for Fig. 1.7*a*:

$$Z_{in}(s)|_{Z_B \to \infty} = (R_1 + R_3) \| (R_2 + R_4) \frac{1 + g_m R_1 \| R_3}{1 + \dfrac{g_m R_1}{1 + (R_1 + R_2)/(R_3 + R_4)}} \tag{1.17}$$

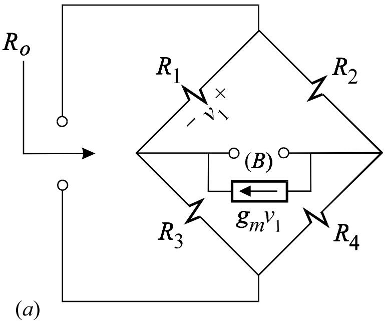

Figure 1.7

To obtain $Z_{in}(s)$, all we need to do is determine $\mathscr{R}^{(B)}$ and $R^{(B)}$, shown in Figs. 1.7*b* and *c*, respectively, and apply the EET:

$$Z_{in}(s) = Z_{in}(s)|_{Z_B \to \infty} \frac{1 + \dfrac{\mathscr{R}^{(B)}}{Z_B}}{1 + \dfrac{R^{(B)}}{Z_B}} \tag{1.18}$$

$$= R_o \frac{1 + sC_B \mathscr{R}^{(B)}}{1 + sC_B R^{(B)}}$$

in which $R_o = Z_{in}(s)|_{Z_B \to \infty}$ and is given by Eq. (1.17).

In Fig. 1.7*b*, the current i_T is given by the sum of $g_m v_1$ and the current through the branch $R_1 \| R_3 + R_2 \| R_4$, so that we have:

$$i_T = g_m v_1 + \frac{v_T}{R_1 \| R_3 + R_2 \| R_4} \tag{1.19}$$

In Fig. 1.7*b* we can also see that:

$$v_1 = v_T \frac{R_1 \| R_3}{R_1 \| R_3 + R_2 \| R_4} \tag{1.20}$$

Substituting Eq. (1.20) in (1.19), we obtain:

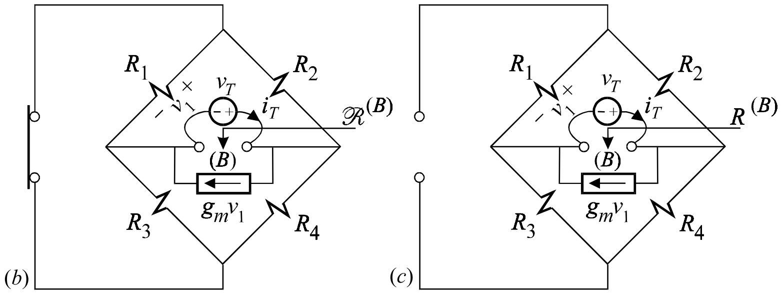

Figure 1.7 (*cont.*)

$$\mathscr{R}^{(B)} = \frac{v_T}{i_T} = \frac{R_1 \parallel R_3 + R_2 \parallel R_4}{1 + g_m R_1 \parallel R_3} \tag{1.21}$$

In Fig. 1.7*c*, the current i_T consists of the sum of $g_m v_1$ and the current through the branches $(R_1 + R_2)$ and $(R_3 + R_4)$ so that we have:

$$i_T = g_m v_1 + \frac{v_T}{R_1 + R_2} + \frac{v_T}{R_3 + R_4} \tag{1.22}$$

In Fig. 1.7*c* we can also see that:

$$v_1 = v_T \frac{R_1}{R_1 + R_2} \tag{1.23}$$

Substituting Eq. (1.23) in (1.22) we obtain:

$$i_T = \frac{v_T(g_m R_1 + 1)}{R_1 + R_2} + \frac{v_T}{R_3 + R_4} \tag{1.24}$$

whence it follows that:

$$R^{(B)} = \frac{v_T}{i_T} = \frac{R_1 + R_2}{1 + g_m R_1} \Bigg\| (R_3 + R_4) \tag{1.25}$$

With $\mathscr{R}^{(B)}$ and $R^{(B)}$ determined, we can write $Z_{in}(s)$ in Eq. (1.18) in pole-zero form:

$$Z_{in}(s) = R_o \frac{1 + s/\omega_z}{1 + s/\omega_p} \tag{1.26}$$

in which:

$$\omega_z = \frac{1}{C_B \mathscr{R}^{(B)}} = \frac{1 + g_m R_1 \parallel R_3}{C_B(R_1 \parallel R_3 + R_2 \parallel R_4)} \tag{1.27}$$

$$\omega_p = \frac{1}{C_B R^{(B)}} = \frac{1}{C_B \dfrac{R_1 + R_2}{1 + g_m R_1} \Bigg\| (R_3 + R_4)} \tag{1.28}$$

And such are the joys of network analysis!

1.5 Review

Although the matrix algebra of nodal or loop analysis is useful in obtaining numerical solutions of linear electrical circuits, it is not useful in obtaining *meaningful* analytical results in symbolic form. An analytical answer is not a mere collection of symbols but an answer in which the symbols are arranged in useful, or low-entropy, forms such as series-parallel combinations and ratios of various elements and time constants. This book presents efficient analytical tools for fast derivation of low-entropy results for electrical circuits. One such analytical tool is the extra element theorem (EET) which we have introduced in this chapter by way of examples in which the input impedance of various bridge circuits is determined.

Problems

1.1 High entropy versus low entropy. In order to appreciate the difference between high- and low-entropy expressions, consider the following for the input impedance of the circuit in the black box:

$$R_{in} = \frac{R_4 R_1 R_2 + R_4 R_1 R_3 + R_4 R_2 R_3}{R_4 R_2 + R_3 R_4 + R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{1.29}$$

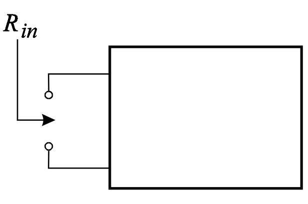

Figure 1.8

Are you able to make anything out of this expression? How does this expression simplify if $R_2 \ll R_3$? Consider now:

$$R_{in} = R_4 \parallel (R_1 + R_2 \parallel R_3) \tag{1.30}$$

Show that the two expressions above are equivalent. Which of the two is more meaningful? Using Eq. (1.30) show that when $R_2 \ll R_3$, we have the following simplification:

$$R_{in} \approx R_4 \parallel (R_1 + R_2) \tag{1.31}$$

1.2 Impedance using the EET. Following the example in Section 1.2, show in a few steps that the input impedance of the circuit below is given by:

$$Z_{in} = R_o \frac{1 + s/\omega_1}{1 + s/\omega_2} \tag{1.32}$$

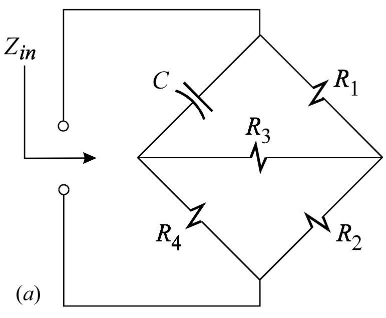

Figure 1.9

where:

$$\left.\begin{aligned} R_o &= R_1 + R_2 \parallel (R_3 + R_4) \\ \omega_1 &= \frac{1}{CR_4 \parallel (R_3 + R_1 \parallel R_2)} \\ \omega_2 &= \frac{1}{C[R_1 + R_3 \parallel (R_4 + R_2)]} \end{aligned}\right\} \quad (1.33a, b, c)$$

Hint: Refer to Figs. 1.9*b*, *c* and *d* below and apply the EET in Eq. (1.4).

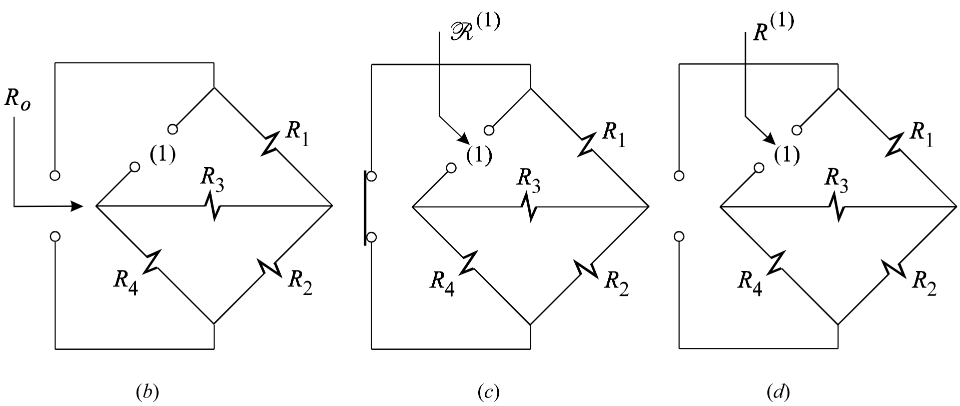

Figure 1.9 (*cont.*)

1.3 Output resistance of a current source using the EET. Show that the output resistance of the BJT current source in Fig. 1.10*a*, using the equivalent circuit model in Fig. 1.10*b*, is given by:

$$R_{out} = \frac{r_\mu + R_s}{1 + \dfrac{R_s}{R_E}} \frac{1 + \dfrac{1}{g_m r_\pi}\left(1 + \dfrac{r_\pi + R_s \parallel r_\mu}{R_E \parallel r_o}\right)}{1 + \dfrac{1}{g_m r_o}\left[1 + \dfrac{r_\mu + r_o}{r_\pi \parallel (R_s + R_E)}\right]} \tag{1.34}$$

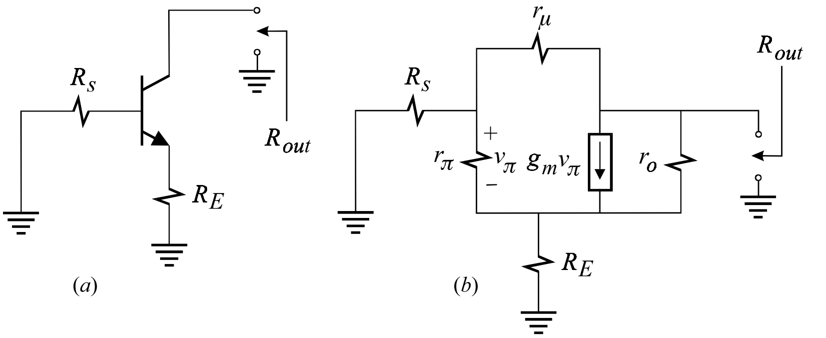

Figure 1.10

Hint: Refer to the example in Section 1.3 and to Figs. 1.10*c*–*e* below.

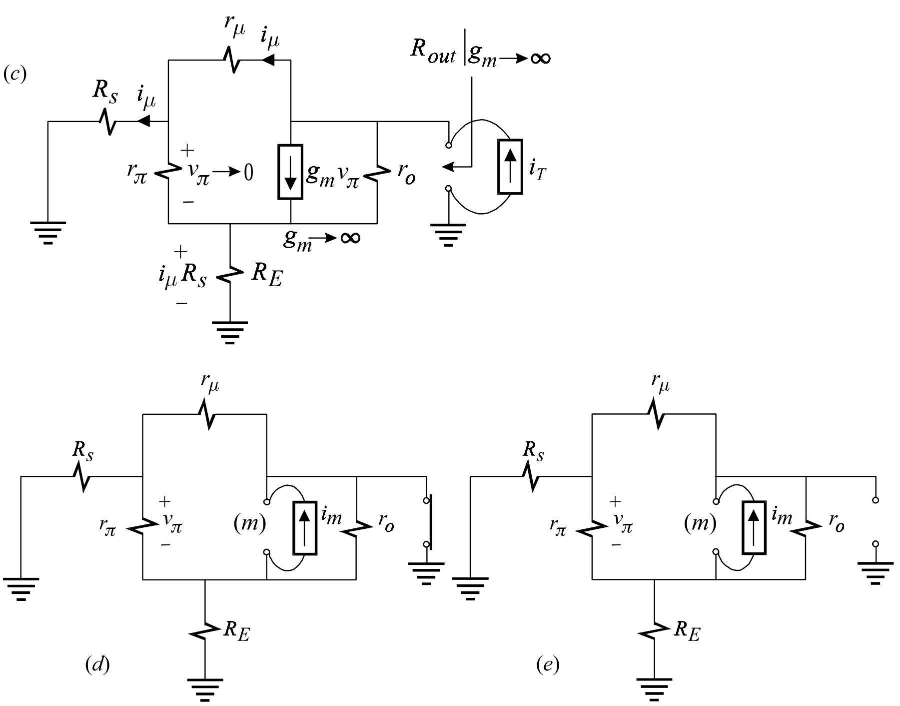

Figure 1.10 (*cont.*)

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