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1

Introduction to Modern Methods for Classical and Quantum Fields in General Relativity

Thierry Daudé, Dietrich Häfner and Jean-Philippe Nicolas

The last few decades have seen major developments in asymptotic analysis in the framework of general relativity, with the emergence of methods that, until recently, were not applied to curved Lorentzian geometries. This has led notably to the proof of the stability of the Kerr–de Sitter spacetime by P. Hintz and A. Vasy [17]. An essential feature of many recent works in the field is the use of dispersive estimates; they are at the core of most stability results and are also crucial for the construction of quantum states in quantum field theory, domains that have a priori little in common. Such estimates are in general obtained through geometric energy estimates (also referred to as vector field methods) or via microlocal/spectral analysis. In our minds, the two approaches should be regarded as complementary, and this is a message we hope this volume will convey succesfully. More generally than dispersive estimates, asymptotic analysis is concerned with establishing scattering-type results. Another fundamental example of such results is asymptotic completeness, which, in many cases, can be translated in terms of conformal geometry as the well-posedness of a characteristic Cauchy problem (Goursat problem) at null infinity. This has been used to develop alternative approaches to scattering theory via conformal compactifications (see for instance F. G. Friedlander [11] and L. Mason and J.-P. Nicolas [22]). The presence of symmetries in the geometrical background can be a tremendous help in proving scattering results, dispersive estimates in particular. What we mean by symmetry is generally the existence of an isometry associated with the flow of a Killing vector field, though there exists a more subtle type of symmetry, described sometimes as hidden, corresponding to the presence of Killing spinors for instance. Recently, the vector field method has been adapted to take such generalized symmetries into account by L. Andersson and P. Blue in [2].

This volume compiles notes from the eight-hour mini-courses given at the summer school on asymptotic analysis in general relativity, held at the Institut

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Fourier in Grenoble, France, from 16 June to 4 July 2014. The purpose of the summer school was to draw an up-to-date panorama of the new techniques that have influenced the asymptotic analysis of classical and quantum fields in general relativity in recent years. It consisted of five mini-courses:

- "Geometry of black hole spacetimes" by Lars Andersson, Albert Einstein Institut, Golm, Germany;
- "An introduction to quantum field theory on curved spacetimes" by Christian Gérard, Paris 11 University, Orsay, France;
- "An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity" by Rod Gover, Auckland University, New Zealand;
- "The bounded L^2 conjecture" by Jérémie Szeftel, Paris 6 University, France;
- "A minicourse on microlocal analysis for wave propagation" by András Vasy, Stanford University, United States of America.

Among these, only four are featured in this book. The proof of the bounded L^2 conjecture having already appeared in two different forms [20, 21], Jérémie Szeftel preferred not to add yet another version of this result; his lecture notes are therefore not included in the present volume.

1.1. Geometry of Black Hole Spacetimes

The notion of a black hole dates back to the 18th century with the works of Simpson and Laplace, but it found its modern description within the framework of general relativity. In fact the year after the publication of the general theory of relativity by Einstein, Karl Schwarzschild [30] found an explicit non-trivial solution of the Einstein equations that was later understood to describe a universe containing nothing but an eternal spherical black hole. The Kerr solution appeared in 1963 [19] and, with the singularity theorems of Hawking and Penrose [15], black holes were eventually understood as inevitable dynamical features of the evolution of the universe rather than mere mathematical oddities. The way exact black hole solutions of the Einstein equations were discovered was by imposing symmetries. First Schwarzschild looked for spherically symmetric and static solutions in four spacetime dimensions, which reduces the Einstein equations to a non-linear ordinary differential equation (ODE). The Kerr solution appears when one relaxes one of the symmetries and looks for stationary and axially symmetric solutions. Roy Kerr obtained his solution by imposing on the metric the so-called "Kerr–Schild" ansatz that corresponds to assuming a special algebraic property for

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the Weyl tensor, namely that it has Petrov-type D, which is similar to the condition for a polynomial to have two double roots. This algebraic speciality of the Weyl tensor can be understood as another type of symmetry assumption about spacetime. This is a generalized symmetry that does not correspond to an isometry generated by the flow of a vector field, but is related to the existence of a Killing spinor. The Kerr family, which contains Schwarzschild's spacetime as the zero angular momentum case, is expected to be the unique family of asymptotically flat and stationary (perhaps pseudo-stationary, or locally stationary, would be more appropriate) black hole solutions of the Einstein vacuum equations (there is a vast literature on this topic, see for example the original paper by D. Robinson [27], his review article [28] and the recent analytic approach by S. Alexakis, A. D. Ionescu, and S. Klainerman [1]). Moreover it is believed to be stable (there is also an important literature on this question, the stability of Kerr–de Sitter black holes was established recently in [17], though the stability of the Kerr metric is still an open problem). These two conjectures play a crucial role in physics where it is commonly assumed that the long term dynamics of a black hole stabilizes to a Kerr solution. The extended lecture notes by Lars Andersson, Thomas Bäckdahl, and Pieter Blue take us through the many topics that are relevant to the questions of stability and uniqueness of the Kerr metric, including the geometry of stationary and dynamical black holes with a particular emphasis on the special features of the Kerr metric, spin geometry, dispersive estimates for hyperbolic equations and generalized symmetry operators. The type D structure is an essential focus of the course, with the intimate links between the principal null directions, the Killing spinor, Killing vectors and tensors, Killing–Yano tensors and symmetry operators. All these notions are used in the final sections where some conservation laws are derived for the Teukolsky system governing the evolution of spin $n/2$ zero rest-mass fields, and a new proof of a Morawetz estimate for Maxwell fields on the Schwarzschild metric is given.

1.2. Quantum Field Theory on Curved Spacetimes

In the 1980s, Dimock and Kay started a research program concerning scattering theory for classical and quantum fields on the Schwarzschild spacetime; see [9]. Their work was then pushed further by Bachelot, Häfner, and others, leading in particular to a mathematically rigorous description of the Hawking effect on Schwarzschild and Kerr spacetimes, see e.g. [4], [14]. In the Schwarzschild case there exists a global timelike Killing vector field in the exterior of the black hole that can be used to define vacuum and thermal states.

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However, it is not clear how to extend these states to the whole spacetime. From a more conceptual point of view this is also quite unsatisfactory because the construction of vacuum states on the Minkowski spacetime uses the full Poincaré group. In addition general spacetimes will not even be locally stationary. On a curved spacetime, vacuum states are therefore replaced by so-called Hadamard states. These Hadamard states were first characterized by properties of their two-point functions, which had to have a specific asymptotic expansion near the diagonal. In 1995 Radzikowski reformulated the old Hadamard condition in terms of the wave front set of the two-point function; see [26]. Since then, microlocal analysis has played an important role in quantum field theory in curved spacetime, see e.g. the construction of Hadamard states using pseudodifferential calculus by Gérard and Wrochna [13]. The lectures given by Christian Gérard give an introduction to quantum field theory on curved spacetimes and in particular to the construction of Hadamard states.

1.3. Conformal Geometry and Conformal Tractor Calculus

Conformal compactifications were initially used in general relativity by André Lichnerowicz for the study of the constraints. It is Roger Penrose who started applying this technique to Lorentzian manifolds, more specifically to asymptotically flat spacetimes, in the early 1960s (see Penrose [25]). The purpose was to replace complicated asymptotic analysis by simple and natural geometrical constructions. To be precise, a conformal compactification allows one to describe infinity for a spacetime $(\mathcal{M}, g)$ as a finite boundary for the manifold $\mathcal{M}$ equipped with a well-chosen metric $\hat{g}$ that is conformally related to g. Provided a field equation has a suitably simple transformation law under conformal rescalings, ideally conformal invariance or at least some conformal covariance, the asymptotic behavior of the field on $(\mathcal{M}, g)$ can be inferred from the local properties at the boundary of the conformally rescaled field on $(\mathcal{M}, \hat{g})$. Penrose's immediate goal was to give a simple reformulation of the Sachs peeling property as the continuity at the conformal boundary of the rescaled field. But he had a longer term motivation which was to construct a conformal scattering theory for general relativity, allowing the setting of data for the spacetime at its past null conformal boundary and to propagate the associated solution of the Einstein equations right up to its future null conformal boundary. Since its introduction, the conformal technique has been used to prove global existence for the Einstein equations, or other non-linear hyperbolic equations, for sufficiently small data (see for example Y. Choquet-Bruhat

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and J. W. York [8]), to construct scattering theories for linear and non-linear test fields, initially on static backgrounds and, in recent years, in time dependent situations and on black hole spacetimes (see L. Mason and J.-P. Nicolas [22] and Nicolas [24] and references therein). It has also been applied to spacetimes with a non-zero cosmological constant. There is an important literature from the schools of R. Mazzeo and R. Melrose and more recently numerous studies using the tractor calculus approach by A. R. Gover and his collaborators. Tractor calculus in its conformal version started from the notion of a local twistor bundle on four-dimensional spin-manifolds as an associated bundle to the Cartan conformal connection, though it in fact dates back to T. Y. Thomas's work [31]. The theory in its modern form first appeared in the founding paper by T. Bailey, M. Eastwood, and Gover [6] where its origins are also thoroughly detailed. The extended lecture notes by Sean Curry and Rod Gover give an up-to-date presentation of the conformal tractor calculus: the first four lectures are mainly focused on the search for invariants; the second half of the course uses tractor calculus to study conformally compact manifolds with application to general relativity as its main motivation.

1.4. A Minicourse in Microlocal Analysis and Wave Propagation

One of the central questions in mathematical relativity is the stability of the Kerr or the Kerr–de Sitter spacetime. As mentioned above, stability has been established by Hintz and Vasy for the Kerr–de Sitter metric, and the question remains open for the Kerr metric. The advantage of the Kerr–de Sitter case is that the inverse of the Fourier transformed d'Alembert operator has a meromorphic extension across the real axis in appropriate weighted spaces. The poles of this extension are then called resonances. Resonances in general relativity were first studied from a mathematical point of view by Bachelot and Motet-Bachelot in [5]. Bony and Häfner gave a resonance expansion of the local propagator for the wave equation on the Schwarzschild–de Sitter metric [7] using the localization of resonances by Sá Barreto-Zworski [29]. Then Dyatlov, Hintz, Vasy, Wunsch, and Zworski made new progress leading eventually to a resonance expansion for the wave equation on spacetimes which are perturbations of the Kerr–de Sitter metric; see the work of Vasy [32]. The whole program culminated in the proof of the non-linear stability of the Kerr–de Sitter metric by Hintz and Vasy [17]. Many aspects come into this study. The first is trapping. Trapping situations were studied in the 1980s for the wave equation outside two obstacles by Ikawa who obtained local energy decay with

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loss of derivatives in this situation; see [18]. The trapping that appears on the Kerr (or the Kerr–de Sitter) metric is r-normally hyperbolic at least for small angular momentum. Suitable resolvent estimates for this kind of situation have been shown by Wunsch–Zworski [33] and Dyatlov [10]. Another important aspect is the presence of supperradiance due to the fact that there is no globally timelike Killing field outside a Kerr–de Sitter black hole. Whereas the cut-off resolvent can nevertheless be extended meromorphically across the real axis using the work of Mazzeo–Melrose [23] and several different Killing fields (see [12]), a more powerful tool to obtain suitable estimates is the Fredholm theory for non-elliptic settings developed by Vasy [32]. Microlocal analysis was first developed for linear problems. Nevertheless, as the work of Hintz–Vasy shows strikingly enough, it is also well adapted to quasilinear problems. In this context one needs to generalize some of the important theorems (such as the propagation of singularities) to very rough metrics. This program has been achieved by Hintz; see [16]. The last important aspect in the proof of the non-linear stability of the Kerr–de Sitter metric is the issue of the gauge freedom in the Einstein equations. Roughly speaking, a linearization of the Einstein equations can create resonances whose imaginary parts have the "bad sign," leading to exponentially growing modes. These resonances turn out to be "pure gauge" and can therefore be eliminated by an adequate choice of gauge; see [17]. The lectures notes by András Vasy introduce the essential tools used in the proof of the non-linear stability of the Kerr–de Sitter metric.

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