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Excerpt
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1 Oscillations and Waves

1.1 Introduction

Objects subjected to restoring forces when displaced from their normal positions and released, perform to and from or vibrating motions. They move back and forth along a path, repeating over and over again, a series of motions. Such motion of constant frequency is called periodic motion or harmonic motion and objects performing such type of motion are called harmonic oscillators. In this book, it is tacitly assumed that there is a linear relationship between force and displacement; frequency remains constant throughout the motion. In real systems however, the linear behavior, implicit in simple harmonic motion, is rarely obeyed. If the frequency of the oscillatory system is not constant, then it is called anharmonic motion – its study is beyond the scope of this book due to its mathematical complexities.

In oscillatory systems, it is not necessarily the bodies themselves who execute oscillations; bodies may be at rest. If the physical properties of a system undergo changes in an oscillatory manner, the system will also be called an oscillatory system. The electromagnetic energy transfer between the capacitor and inductor in a tank circuit used in electronic gadgets, variation of pressure in air due to propagation of sound waves, vibration of the diaphragm of a speaker in sound systems, flow of alternating current, variation of electric and magnetic vectors during propagation of electromagnetic waves, etc., are examples of oscillatory systems.

1.1.1 Parameters of an oscillatory system

i. *Mean position* The position of the oscillating body when there is no oscillation is called the mean position or equilibrium position. This is the rest position of the oscillating body.

ii. *Amplitude* (r) It is the absolute value of the maximum displacement of the oscillating particle from its mean position or equilibrium position.

iii. *Time period* (T) It is the time required for one complete oscillation.

iv. *Frequency* (ν) It is the number of complete oscillations made by the oscillating body in one second. The relation between frequency 'ν' and time period 'T' from definition is $T = \frac{1}{\nu}$.

1.2 Simple Harmonic Oscillation (SHO)

Let a body of mass 'm' be placed on a frictionless plane with a massless spring attached to it (Fig. 1.1). The other end of the spring is fixed to a rigid support. The spring–body system is in the relaxed state, i.e., the spring is neither compressed nor extended. Notice the position of the body – it is called the mean position or equilibrium position. Now, the body is pulled through a displacement 'x'. The spring exerts a restoring force on the body tending to pull it backwards. This restoring force 'F' is proportional to the displacement (i.e., elongation of the spring) 'x' and is opposite in direction to the displacement.

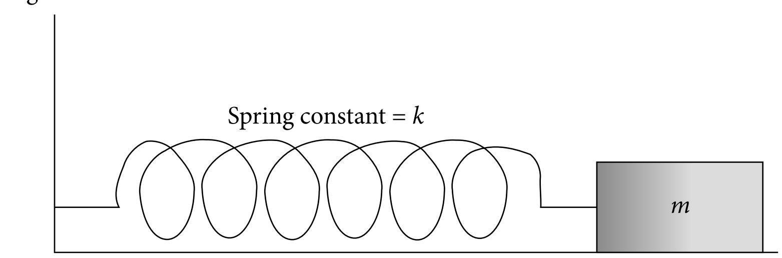

Figure 1.1 | Simple harmonic oscillator. Spring–body system is placed on a frictionless plane

Mathematically, $F \propto -x$ (negative sign appears since the restoring force and displacement 'x' are in opposite directions)

$$F = -kx \tag{1.1}$$

where k is the proportionality constant and is known as spring constant. This equation is called Hooke's law of elasticity. Applying Newton's laws of motion, Hooke's law can be written as

$$ma = -kx$$

or $$m\frac{d^2x}{dt^2} = -kx \tag{1.2}$$

Equation (1.2) is the differential equation of motion of a simple harmonic oscillator in the absence of other forces. It can also be written as

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{1.3}$$

where $\omega_0^2 = \dfrac{k}{m}$ (1.4)

Here m is the mass of the body attached at the end of the free end of the spring and ω_0 is called the natural angular frequency of oscillation. The general solution of the differential Eq. (1.3) is determined in the following way. Other methods for the purpose are also available.

Assume a solution of the form

$$x = e^{pt} \tag{1.5}$$

Putting (1.5) into (1.3), we get

$$p^2 e^{pt} + \omega_0^2 e^{pt} = 0$$

or $p = \pm i\omega_0$

The general solution of the differential Eq. (1.3) will be given by

$$x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \tag{1.6}$$

The constants C_1 and C_2 must be complex in order for Eq. (1.6) to be the general solution. Since $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ are complex conjugates of each other, the constants C_1 and C_2 must be complex conjugate of each other so that x is real. For this, we set

$$C_1 = C = Ae^{i\theta}$$

and $C_2 = C^* = Ae^{-i\theta}$

Upon this substitution into Eq. (1.6), we get

$$x = Ce^{i\omega_0 t} + C^* e^{-i\omega_0 t} = Ae^{i\theta} e^{i\omega_0 t} + Ae^{-i\theta} e^{-i\omega_0 t}$$

or $x = 2A\cos(\omega_0 t + \theta) = r\cos(\omega_0 t + \theta),\ 2A = r$ (1.7)

($x = r\sin(\omega_0 t + \theta)$ may also be a solution).

Here

r = amplitude of the oscillation.

θ = the initial phase of the oscillation.

$\omega_0 t + \theta$ = phase of the motion.

The constants r and θ can be obtained from the initial conditions of the simple harmonic motion.

The velocity 'v' of the body executing SHO is determined by differentiating Eq. (1.7) with respect to time.

$$v = -r\omega_0 \sin(\omega_0 t + \theta) \tag{1.8}$$

The acceleration 'a' of the body executing SHO is determined by differentiating Eq. (1.8) with respect to time.

$$a = -\omega_0^2 r \cos(\omega_0 t + \theta). \tag{1.9}$$

Putting the value of $r\cos(\omega_0 t + \theta) = x$ from (1.7) into Eq. (1.9), we get

$$a = -\omega_0^2 x \tag{1.10}$$

Equation (1.10) shows that in the case of a simple harmonic motion, acceleration is directed opposite to displacement.

If T is the time period, then in T seconds, the number of complete oscillations is 1. So, in 1 second, the number of oscillations will be $1/T$ which by definition is the frequency ν. Hence, we have

$$\nu = \frac{1}{T} \tag{1.11}$$

Frequency and time period are inversely proportional to each other.

From the definition of angular frequency, we have

$$\omega_0 = \frac{2\pi}{T} = 2\pi\nu \tag{1.12}$$

1.2.1 Energy of a simple harmonic oscillator

By the definition of potential energy, the potential energy of a simple harmonic oscillator at any instant or at any position is given by

$$E_P = \int_0^x F dx = \int_0^x kx dx = \frac{1}{2}kx^2 \tag{1.13}$$

Here 'x' has been measured from the mean position x = 0. Putting the value of x from Eq. (1.7) into Eq. (1.13), we have

$$E_P = \frac{1}{2}kr^2\cos^2(\omega_0 t + \theta) \tag{1.14}$$

Putting the value of k from Eq. (1.4) into Eq. (1.14), we get

$$E_P = \frac{1}{2}m\omega_0^2 r^2\cos^2(\omega_0 t + \theta) \tag{1.15}$$

Since the maximum value of $\cos^2(\omega_0 t + \theta)$ is 1, the maximum value of potential energy-$E_{P\max}$ from Eq. (1.15) is found out to be

$$E_{P\max} = \frac{1}{2}m\omega_0^2 r^2 \tag{1.16}$$

By the definition of kinetic energy, the kinetic energy of a simple harmonic oscillator at any instant or at any position is given by

$$E_K = \frac{1}{2}mv^2$$

Putting the value of v from Eq. (1.8) into the previous equation, we get

$$E_K = \frac{1}{2}mr^2\omega_0^2\sin^2(\omega_0 t + \theta) \tag{1.17}$$

Since the maximum value of $\sin^2(\omega_0 t + \theta)$ is 1, the maximum value of kinetic energy $E_{K\max}$ from Eq. (1.17) is found out to be

$$E_{K\max} = \frac{1}{2}m\omega_0^2 r^2 \tag{1.18}$$

The total energy of a simple harmonic oscillator at any instant or at any position is given by

$$E = E_P + E_K$$

$$= \frac{1}{2}m\omega_0^2 r^2\cos^2(\omega_0 t + \theta) + \frac{1}{2}mr^2\omega_0^2\sin^2(\omega_0 t + \theta)$$

$$= \frac{1}{2}m\omega_0^2 r^2 = E_{K\max} = E_{P\max} \tag{1.19}$$

Equation (1.19) shows that the total energy of a simple harmonic oscillator at any instant or at any position is constant.

1.2.2 Characteristics of SHO

From the discussions in the previous section, we can infer the following characteristics of SHO.

When the body is released, it moves to and fro about the mean position with constant amplitude '*r*'. As the body moves towards the mean position, its speed increases but the force and hence, its acceleration decreases; both become zero at the mean position. Due to the inertia of motion, the body overshoots the mean position, but at the same time, a retarding force comes into action to oppose the motion. This retarding force increases until the body reaches the largest distance from the mean position. Here, it stops and begins its return journey. This process goes on continuously for an indefinite period if there is no dissipative force. Throughout the motion, the force as well as the acceleration is directly proportional to displacement and directed towards the mean position. The distances travelled by the vibrating body on the two sides of the mean position are equal. This type of motion is called simple harmonic oscillation. The SHO is graphically represented in Fig. 1.2.

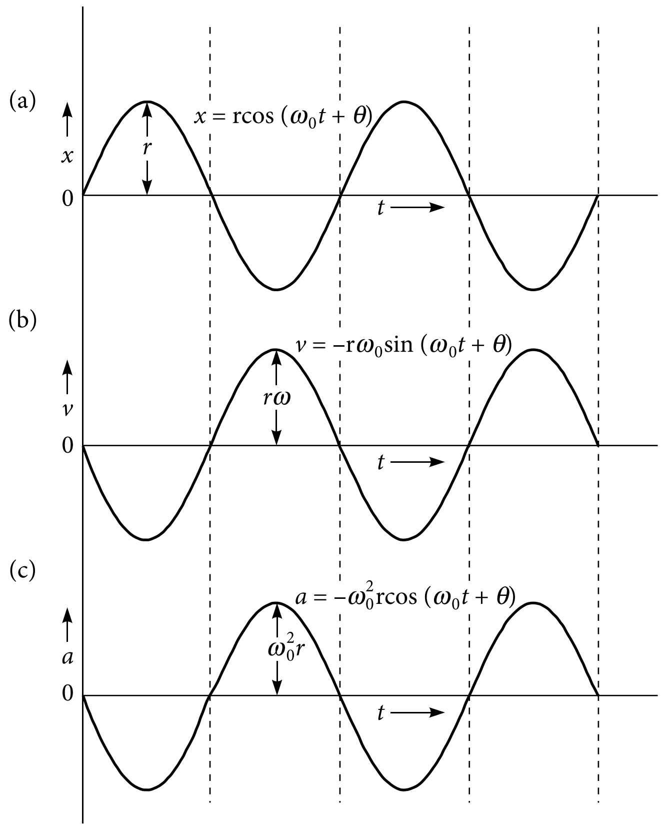

Figure 1.2 Graphical representation of SHO. (a) displacement-time plot, (b) velocity–time plot, (c) acceleration–time plot

For a particular simple harmonic oscillator, mass, amplitude and angular frequency are constants with respect to time. Therefore, we can conclude that the total energy of a simple harmonic oscillator at any instant or position is constant. Moreover, the total energy of a simple harmonic oscillator at any instant or position is equal to the maximum values of potential or kinetic energy. The simultaneous variation of potential energy and kinetic energy of a simple harmonic oscillator according to Eqs (1.15) and (1.17) is depicted in Fig. 1.3(a) and in Fig. 1.3(b).

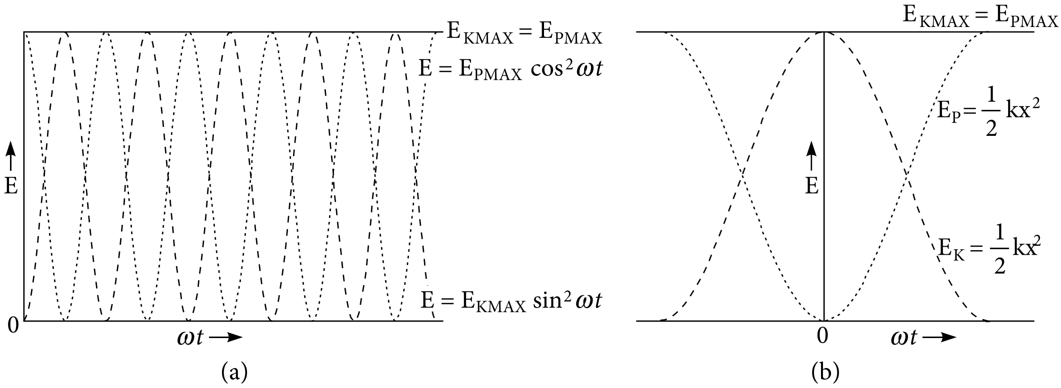

Figure 1.3 Energy of a simple harmonic oscillator. The dotted curve represents potential energy, the dashed line curve represents kinetic energy and the continuous curve represents the total energy of a simple harmonic oscillator

We will now cite few examples of simple harmonic motion under ideal conditions. The derivations for time periods in all the examples are left as exercises to the students.

i. The motion of a simple pendulum in vacuum is simple harmonic. Its time period is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}}, \quad \ell = \text{length of the simple pendulum.}$$

ii. Take a cleaned U-shaped glass tube and fix it vertically on a stand. Partially fill it with mercury. Now the levels of the mercury column in both sides are equal. Blow slowly through one end so that the mercury column in the other end rises slightly. When we stop blowing, the mercury column executes simple harmonic motion having time period

$$T = 2\pi\sqrt{\frac{\ell}{2g}}, \quad \ell = \text{length of the mercury column in the U-tube.}$$

iii. A body dropped into a hole dug through the centre of earth executes simple harmonic motion with time period

$$T = 2\pi\sqrt{\frac{R}{g}},\ R = \text{radius of Earth.}$$

iv. Place a small block inside a smooth curved surface having radius of curvature 'R'. Move the sphere slightly along the curved surface away from the equilibrium position and then release. It will perform simple harmonic motion with time period

$$T = 2\pi\sqrt{\frac{R}{g}}.$$

v. Place a sphere of radius 'r' inside a curved surface of radius of curvature 'R'. Move the sphere slightly along the curved surface away from the equilibrium position and then release. It will execute simple harmonic motion with time period

$$T = 2\pi\sqrt{\frac{7(R-r)}{5g}} \approx 2\pi\sqrt{\frac{7R}{5g}} \quad \text{if} \quad R >> r$$

vi. If a small iron cylinder, partially submerged in mercury vertically, is pressed slightly and then released, it will execute simple harmonic motion with time period

$T = 2\pi\sqrt{\frac{\ell\rho}{\rho' g}}$. Here ℓ = length of the cylinder, ρ = density of the cylinder, ρ' = density of mercury.

Example 1.1

A spring 15 cm in length is fixed to the ceiling. When a body of mass 1 kg is hung at the free end, its length becomes 17 cm. Calculate the spring constant of the spring.

Solution

The increase in length 'x' of the spring when a 1kg body is hung = 17 cm – 15 cm = 2 cm = 0.02 m

The downward force acting on the spring = weight of the body hung. At the equilibrium position, the restoring force = downward force acting on the spring.

$$kx = mg$$

$$\text{or} \quad k = \frac{mg}{x} = \frac{1 \times 9.8 \text{ N}}{0.02 \text{ m}} = 490 \text{ N/m}$$

The spring constant of the spring is calculated to be 490 N/m.

Example 1.2

A 5 kg body extends a spring 15 cm from its relaxed position. The body is removed and a 1 kg body is hung from the same spring. The 1 kg body is pulled and released. Calculate the time period of oscillation of the body.

Solution

The increase in length 'x' of the spring when a 5 kg body is hung = 15 cm = 0.15 m

The downward force acting on the spring = weight of the body hung. At the equilibrium position, the restoring force = downward force acting on the spring.

$$kx = mg$$

or $k = \dfrac{mg}{x} = \dfrac{5 \times 9.8\ \text{N}}{0.15\ \text{m}} = 326.67\ \text{N/m}$ is the spring constant of the spring.

The natural angular frequency of the oscillation $\omega_0 = \sqrt{\dfrac{k}{m}}$.

The time period of oscillation $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{1}{326.67}}\ \text{s} = 0.35\ \text{s}$

Example 1.3

A body oscillates with simple harmonic motion obeying the equation $y = 12\ \cos\left(0.7\pi t + \dfrac{\pi}{5}\right)$m.

Calculate the velocity and acceleration of the body at time t = 3 s. Also calculate the natural frequency and time period of the harmonic motion.

Solution

The velocity 'v' of the body at any time t is

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[12\cos\left(0.7\pi t + \frac{\pi}{5}\right)\right]\text{m/s} = -8.4\pi\sin\left(0.7\pi \text{t} + \frac{\pi}{5}\right)\text{m/s}$$

So the velocity of the body at t = 3 s will be

$$v = -8.4\pi\sin\left(0.7\pi \times 3 + \frac{\pi}{5}\right)\ \text{m/s} = -21.35\ \text{m/s}$$

The velocity is directed opposite to the displacement.

The acceleration 'a' of the body at any time t is

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[-8.4\pi\sin\left(0.7t\pi + \frac{\pi}{5}\right)\right]\text{m/s}^2 = -5.88\pi^2\cos\left(0.7\pi t + \frac{\pi}{5}\right)\text{m/s}^2$$

So the acceleration of the body at t = 3 s will be

$$a = -5.88\pi^2 \cos\left(0.7\pi \times 3 + \frac{\pi}{5}\right) \text{ m/s}^2 = -34.11 \text{ m/s}^2.$$

The acceleration is directed opposite to the displacement.

The initial phase constant $= \dfrac{\pi}{5}$.

The angular frequency $\omega = 0.7\pi \text{s}^{-1}$

Hence, the frequency of the oscillation $\nu = \dfrac{0.7\,\text{s}^{-1}}{2} = 0.35\,\text{s}^{-1}$

The time period of the oscillation $T = \dfrac{1}{\nu} = 2.86\,\text{s}$

1.3 Damped Harmonic Oscillation (DHO)

In case of simple harmonic oscillation, the amplitude of oscillation does not decrease. However in reality, in the case of simple pendulums, amplitude decreases with the passing of time. This is due to the viscosity of the medium. The force is the dissipative force F_d. Harmonic oscillation under the influence of a spring–like restoring force in a viscous medium is called damped harmonic oscillation.

For a low velocity dissipative force, F_d is directly proportional to the velocity of the oscillating body and is always directed opposite to the velocity of the body. Mathematically, we have

$$F_d = -b\frac{dx}{dt} \tag{1.20}$$

The minus sign appears because the direction of the dissipative force F_d is opposite to that of velocity. The restoring force on the oscillating body from Eq. (1.1) is $-kx$. Therefore, the total force on the vibrating body in a dissipative medium is

$$F_{\text{total}} = -b\frac{dx}{dt} - kx \tag{1.21}$$

or $$ma = -b\frac{dx}{dt} - kx$$

or $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \tag{1.22}$$