

Index

This should be used in conjunction with the bibliography and the list of symbols. Numbers following entries are page numbers which, if accompanied by (Pr $n.m$), refer to Problem $n.m$ on that page; a number with a trailing ‘ n ’ indicates that a footnote is being referenced. Unless stated otherwise ‘integral’, ‘integrability’, etc. always refer to the (abstract) Lebesgue integral. Within the index we use ‘L-...’ and ‘R-...’ as a shorthand for ‘(abstract) Lebesgue-...’ and ‘Riemann-...’.

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