MEASURES, INTEGRALS AND MARTINGALES

A concise yet elementary introduction to measure and integration theory, which are vital in many areas of mathematics, including analysis, probability, mathematical physics and finance. In this highly successful textbook the core ideas of measure and integration are explored, and martingales are used to develop the theory further. Additional topics are also covered such as: Jacobi's transformation theorem; the Radon–Nikodym theorem; differentiation of measures and Hardy–Littlewood maximal functions.

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Contents

List of Symbols

This is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed. Some symbols are used locally, without ambiguity, in senses other than those given below. Numbers following entries are page numbers, with the occasional (Pr *m.n*) referring to Problem *m.n* on the respective page.

Unless stated otherwise, binary operations between functions such as $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$, comparisons $f \leqslant g$, $f < g$ or limiting relations $f_n \xrightarrow[n\to\infty]{} f$, $\lim_n f_n$, $\liminf_n f_n$, $\limsup f_n$, $\sup_i f_i$ or $\inf_i f_i$ are always understood pointwise.

Alternatives are indicated by square brackets, i.e., '*if A* [*B*] … *then P* [*Q*]' should be read as '*if A* … *then P*' and '*if B* … *then Q*'.

Generalities

positive	always in the sense $\geqslant 0$
negative	always in the sense $\leqslant 0$
$\mathbb{N}$	natural numbers: $1, 2, 3, \dots$
$\mathbb{N}_0$	positive integers: $0, 1, 2, \dots$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integer, rational, real, complex numbers
$\overline{\mathbb{R}}$	$[-\infty, +\infty]$
$\inf\emptyset, \sup\emptyset$	$\inf\emptyset = +\infty$, $\sup\emptyset = -\infty$
$a \vee b$	maximum of a and b
$a \wedge b$	minimum of a and b
$\liminf_n a_n$	$\sup_k \inf_{n\geqslant k} a_n$, 409
$\limsup_n a_n$	$\inf_k \sup_{n\geqslant k} a_n$, 409
$\lvert x\rvert$	Euclidean norm in $\mathbb{R}^n$, $\lvert x\rvert^2 = x_1^2 + \cdots + x_n^2$
$\langle x, y\rangle$	scalar product $\sum_{i=1}^n x_i y_i$
ω_n	volume of the unit ball in $\mathbb{R}^n$, 181

Sets and set operations

$A \cup B$	union 6
$A \,\dot\cup\, B$	union of disjoint sets, 6
$A \cap B$	intersection, 6
$A \setminus B$	set-theoretic difference, 6
A^c	complement of A, 6
$A \vartriangle B$	$(A \setminus B) \cup (B \setminus A)$
$A \subset B$	subset (includes '$=$'), 6
$A \subsetneq B$	proper subset, 6
$A \times B$	Cartesian product
$\overline{A}$	closure of A
A°	open interior of A
$A_n \uparrow A$, $A_n \downarrow A$	23
$A \times B$	Cartesian product
A^n	n-fold Cartesian product
$A^{\mathbb{N}}$	infinite sequences in A
$\#A$	cardinality of A, 8
$t \cdot A$	$\{ta : a \in A\}$
$x + A$	$\{x + a : a \in A\}$
$E \cap \mathscr{A}$	$\{E \cap A : A \in \mathscr{A}\}$, 17
$\liminf_n A_n$	$\bigcup_{k\in\mathbb{N}} \bigcap_{n\geqslant k} A_n$, 413
$\limsup_n A_n$	$\bigcap_{k\in\mathbb{N}} \bigcup_{n\geqslant k} A_n$, 413
$(a, b), [a, b]$	open, closed intervals
$(a, b], [a, b)$	half-open intervals
$B_r(x)$	open ball with radius r and centre x

Families of sets

$\mathscr{A}$	generic σ-algebra, 16
$\overline{\mathscr{A}}$	completion 30 (Pr 4.15)
$(\mathscr{A}_i)_{i\in I}$	filtration, 276
$\mathscr{A}_\infty$	$\sigma(\mathscr{A}_i : i \in I) = \sigma\left(\bigcup_{i\in I} \mathscr{A}_i\right)$
$\mathscr{A}_{-\infty}$	$\bigcap_{\ell\in-\mathbb{N}} \mathscr{A}_\ell$
$\mathscr{A}_\tau$, $\mathscr{A}_\sigma$	283
$\mathscr{A} \times \mathscr{B}$	$\{A \times B : A \in \mathscr{A},\ B \in \mathscr{B}\}$ rectangles
$\mathscr{A} \otimes \mathscr{B}$	product-σ-algebra, 138
$\mathscr{B}(X)$	Borel sets in X, 18: $X = \mathbb{R}^n$ (18), $X = A \subset \mathbb{R}^n$ (22 Pr 3.13), $X = \overline{\mathbb{R}}$ (61), $X = \mathbb{C}$ (88 Pr 10.9, 423)
$\overline{\mathscr{B}(\mathbb{R}^n)}$	completion of the Borel sets, 151 (Pr 14.15), 172, 429

$\mathscr{J}, \mathscr{J}^o, \mathscr{J}_{\mathrm{rat}}$	rectangles in $\mathbb{R}^n$, 19
$\mathscr{N}_\mu$	μ-null sets, 29 (Pr 4.12), 89
$\mathscr{O}(X)$	topology, open sets in X, 18
$\mathscr{P}(X)$	all subsets of X, 13
$\delta(\mathscr{G})$	Dynkin system generated by $\mathscr{G}$, 32
$\sigma(\mathscr{G})$	σ-algebra generated by $\mathscr{G}$, 17
$\sigma(T), \sigma(T_i : i \in I)$	σ-algebra generated by the map(s) T, resp., T_i, 55
$(X, \mathscr{A})$	measurable space, 23
$(X, \mathscr{A}, \mu)$	measure space, 24
$(X, \mathscr{A}, \mathscr{A}_i, \mu)$	filtered measure space, 275, 276

Measures and integrals

μ, ν	generic measures
δ_x	Dirac measure in x, 26
λ, λ^n	Lebesgue measure, 27
$\mu \circ T^{-1}, T(\mu)$	image measure, 55
$u \cdot \mu, u\mu$	measure with density, 86
$\mu \times \nu$	product of measures, 141
$\mu \star \nu$	convolution, 157
$\mu \ll \nu$	absolute continuity, 300
$\mu \perp \nu$	singular measures, 306
$d\nu/d\mu$	Radon–Nikodým derivative, 230, 301
$\mathbb{E}^{\mathscr{G}}$	conditional expectation, 343, 346, 348
$\int u\,d\mu$	74, 82
$\int_A u\,d\mu$	$\int \mathbb{1}_A u\,d\mu$, 86
$\int u\,dx$	83
$\int u\,dT(\mu)$	$\int u \circ T\,d\mu$, 154
$\int_a^b u(x)\,dx$, $(R)\int_a^b u(x)\,dx$	Riemann integral, 102, 443

Functions and spaces

$\mathbb{1}_A$	$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$
$\operatorname{sgn}(x)$	$\mathbb{1}_{(0,\infty)}(x) - \mathbb{1}_{(-\infty,0)}(x)$
id	identity map or matrix
det	determinant (of a matrix)
$u(A)$	$\{u(x) : x \in A\}$
$u^{-1}(\mathscr{B})$	$\{u^{-1}(B) : B \in \mathscr{B}\}$, 17
u^+	$\max\{u(x), 0\}$ positive part
u^-	$-\min\{u(x), 0\}$ negative part
$\{u \in B\}$	$\{x : u(x) \in B\}$, 60
$\{u \geqslant \lambda\}$	$\{x : u(x) \geqslant \lambda\}$ etc., 60
$\operatorname{supp} u$	support $\overline{\{u \neq 0\}}$
$C(X)$	continuous functions on X
$C_b(X)$	bounded continuous functions on X
$C_\infty(X)$	continuous functions on X with $\lim_{\lvert x\rvert\to\infty} u(x) = 0$
$C_c(X)$	continuous functions on X with compact support
$\mathcal{E}(\mathscr{A})$	simple functions, 63
$\mathcal{M}(\mathscr{A})$	measurable functions, 62
$\mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$	measurable functions, values in $\overline{\mathbb{R}}$, 62
$\mathcal{L}^1$	integrable functions, 82
$\mathcal{L}^1_{\overline{\mathbb{R}}}$	integrable functions, values in $\overline{\mathbb{R}}$, 82
$\mathcal{L}^p, \mathcal{L}^\infty$	116
L^p, L^∞	119
$\mathcal{L}^p_{\mathbb{C}}, L^p_{\mathbb{C}}$	423, 214
$\ell^1(\mathbb{N}), \ell^p(\mathbb{N})$	124–125
$\lVert u\rVert_p$	$\left(\int \lvert u\rvert^p\,d\mu\right)^{1/p}$, $p < \infty$, 116
$\lVert u\rVert_\infty$	$\inf\{c : \mu\{\lvert u\rvert \geqslant c\} = 0\}$, 114
$\mathfrak{M}^+_{\mathrm{r}}(X)$	regular measures on X, 437

Abbreviations

a.a., a.e.	almost all/every(where), 89
UI	uniformly integrable, 258
w.r.t.	with respect to
$\cup/\cap$-stable	stable under finite unions/intersections
□	end of proof
[✍]	indicates that a small intermediate step is required
*	can be omitted without loss of continuity
	(in the margin) caution
(D_1)–(D_3)	Dynkin system, 32, 37
(M_0)–(M_3)	measure, 23
(OM_1)–(OM_3)	outer measure, 40, 200
$(\mathcal{O}_1)$–$(\mathcal{O}_3)$	topology, 18
(S_1)–(S_3)	semi-ring, 39
(Σ_1)–(Σ_3)	σ-algebra, 16

Prelude

The purpose of this book is to give a straightforward and yet elementary introduction to measure and integration theory that is within the grasp of second- or third- year undergraduates. Indeed, apart from interest in the subject, the only prerequisites for Chapters 1–15 are a course on rigorous ϵ–δ-analysis on the real line and basic notions of linear algebra and calculus in $\mathbb{R}^n$. The first 15 chapters form a concise introduction to Lebesgue's approach to measure and integration, which I have often taught in 10-week or 30-hour lecture courses at several universities in the UK and Germany.

Chapters 16–28 are more advanced and contain a selection of results from measure theory, probability theory and analysis. This material can be read linearly but it is also possible to select certain topics; see the dependence chart on page xvii. Although these chapters are more challenging than the first part, the prerequisites remain essentially the same and a reader who has worked through and understood Chapters 1–15 will be well prepared for all that follows. I tried to avoid topology and, when it comes in, usually an understanding of an open set and open ball (in $\mathbb{R}^n$) will suffice. From Chapter 17 onwards, I frequently use metric spaces (X, d), but you can safely think of them as $X = \mathbb{R}^n$ and $d(x, y) = |x - y|$ – or read Appendix B.

Each chapter is followed by a section of *problems*. They are not just drill exercises but contain variants of, excursions from and extensions of the material presented in the text. The proofs of the core material do not depend on any of the problems and it is as an exception that I refer to a problem in one of the proofs. Nevertheless, I do advise you to attempt as many problems as possible. The material in the *appendices* – on upper and lower limits, the Riemann integral and tools from topology – is primarily intended as back-up, for when you want to look something up.

Unlike many textbooks this is not an *introduction to integration for analysts* or a *probabilistic measure theory*. I want to reach both (future) analysts and (future) probabilists, and to provide a foundation which will be useful for both communities and for further, more specialized, studies. It goes without saying that I have had to leave out many pet choices of each discipline. On the other hand, I try to intertwine the subjects as far as possible, resulting – mostly in the latter part of the book – in the consequent use of the martingale machinery which gives 'probabilistic' proofs of 'analytic' results.

Measure and integration theory is often seen as an abstract and dry subject, which is disliked by many students. There are several reasons for this. One of them is certainly the fact that measure theory has traditionally been based on a thorough knowledge of real analysis in one and several dimensions. Many excellent textbooks are written for such an audience but today's undergraduates find it increasingly hard to follow such tracts, which are often more aptly labelled *graduate* texts. Another reason lies within the subject: measure theory has come a long way and is, in its modern purist form, stripped of its motivating roots. If, for example, one starts out with the basic definition of measures, it takes unreasonably long until one arrives at interesting examples of measures – the proof of existence and uniqueness of something as basic as Lebesgue measure already needs the full abstract machinery – and it is not easy to entertain by constantly referring to examples made up of delta functions and artificial discrete measures … I try to alleviate this by postulating the existence and properties of Lebesgue measure early on, then justifying the claims as we proceed with the abstract theory.

Technically, measure and integration theory is not more difficult than, say, complex function theory or vector calculus. Most proofs are even shorter and have a very clear structure. The one big exception, Carathéodory's extension theorem, can be safely stated without proof since an understanding of the technique is not really needed at the beginning; we will refer to the details of it only in connection with regularity questions in Chapter 16 and in Chapter 18 on Hausdorff measures. The other exceptions are the Radon–Nikodým theorem (Chapter 20) and the Riesz representation theorem (Chapter 21).

Changes in the second edition The first edition of my textbook was well received by scholars and students alike, and I would like to thank all of them for their comments and positive criticism. There are a few changes to the second edition: while the core material in Chapters 1–15 is only slightly updated, the proof of Jacobi's theorem (Chapter 16) and the material on martingales (Chapters 23 and 24), Hilbert space (Chapter 26) and conditional expectations (Chapter 27) have been re-organized and re-written. Newly added are Chapters 17–21, covering dense and measure-determining sets, Hausdorff measures and Fourier transforms as well as the classical proofs of the Radon–Nikodým and the Riesz representation theorem for L^p and C_c. I hope that these changes do not alter the character of the text and that they will make the book even more useful and accessible.

Acknowledgements I am grateful to the many people who sent me comments and corrections. My colleagues Niels Jacob, Nick Bingham, David Edmunds and Alexei Tyukov read the whole text of the first edition, and Charles Goldie and Alex Sobolev commented on large parts of the manuscript. This second edition was prepared with the help of Franziska Kühn, who did the thankless job of proofreading and suggested many improvements, and Julian Hollender, who produced most of the illustrations. Georg Berschneider and Charles Goldie read the newly added passages. Without their encouragement and help there would be more obscure passages, blunders and typos in the pages to follow. I owe a great debt to all of the students who went to my classes, challenged me to write, re-write and improve this text and who drew my attention – sometimes unbeknownst to them – to many weaknesses.

Figures 1.4, 4.1, 8.1, 8.2, 9.1, 12.2, 13.1 and 21.1 and those appearing on p. 134 and in the proofs of Theorems 3.8 and 8.8 and Lemma 13.1 are taken from my book *Maß und Integral*, De Gruyter, Berlin 2015, ISBN 978-3-11-034814-9. I am grateful for the permission of De Gruyter to use these figures here, too.

Finally, it is a pleasure to acknowledge the interest and skill of Cambridge University Press and its editors, Roger Astley and Clare Dennison, in the preparation of this book.

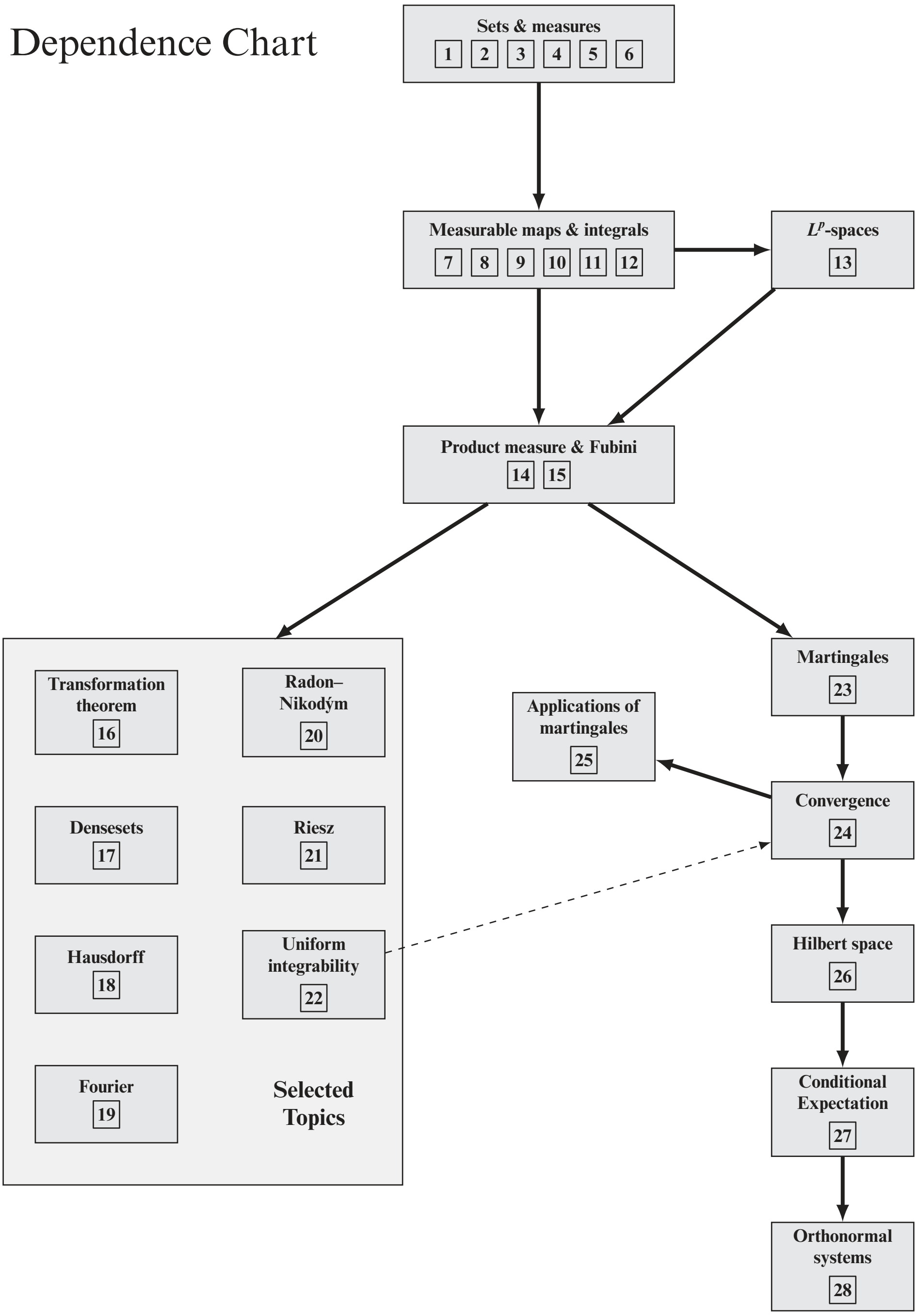
A few words on notation before getting started I try to keep unusual and special notation to a minimum. However, a few remarks are in order: $\mathbb{N}$ means the natural numbers $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. *Positive* or *negative* is always understood in the non-strict sense $\geqslant 0$ or $\leqslant 0$; to exclude 0, I say *strictly positive/negative*. A '+' as sub- or superscript refers to the positive part of a function or the positive members of a set. Finally, $a \vee b$, resp. $a \wedge b$, denote the maximum, resp. minimum, of the numbers $a, b \in \mathbb{R}$. For any other general notation there is a comprehensive list of symbols starting on page x.

In some statements I indicate alternatives using square brackets, i.e. '*if A* [*B*] … *then P* [*Q*]' should be read as '*if A* … *then P*' and '*if B* … *then Q*'. The end of a proof is marked by Halmos' tombstone symbol □, and Bourbaki's dangerous bend symbol in the margin identifies a passage which requires some attention. Throughout Chapters 1–15 I have marked material which can be omitted on first reading without losing (too much) continuity by ✻.

As with every book, one cannot give all the details at every instance. On the other hand, the less experienced reader might glide over these places without even noticing that some extra effort is needed; for these readers – and, I hope, not to the annoyance of all others – I use the symbol [✍] to indicate where a short calculation on the side is appropriate.

Cross-referencing Throughout the text chapters are numbered with arabic numerals and appendices with capital letters. Theorems, definitions, examples, etc. share the same numbering sequence, e.g. Definition 4.1 is followed by Lemma 4.2 and then Corollary 4.3, and $(n.k)$ denotes formula k in Chapter n.

Dependence Chart



Dependence chart Chapters 1–15 contain core material which is needed in all later chapters. The dependence is shown by arrows, with dashed arrows indicating a minor dependence; Chapters 16–22 can be read independently and in any order.