Introduction

This book develops the contents of a series of lectures given at the Institut des Hautes Études Scientifiques in February and March 2015 (see [KS 15]), based on [Ka 84], [DK 13] and [KS 14]. They are addressed to readers familiar with the language of sheaves and D-modules, in the derived sense.

As announced in the title, the subject of this book is holonomic D-modules. The theory of D-modules appeared in the 1970s with the thesis of [Ka 70] and Bernstein's paper [Be 71]. However, already in the 1960s, Mikio Sato had the main ideas of the theory in mind and gave talks at Tokyo University on these topics. Unfortunately, Sato did not write anything and it seems that his ideas were not understood at this time. (See [An 07, Sc 07].)

A left coherent $\mathscr{D}_X$-module on a complex manifold X is locally represented by (the cokernel of) a matrix of differential operators acting on the right. Hence, D-module theory is essentially the algebraic study of systems of linear partial differential equations. It seems that algebraic geometers were frightened by the non-commutative nature of the sheaf of rings $\mathscr{D}_X$, and it may be the reason why one had to wait untill the 1970s until the theory appeared. But once one realizes that the ring $\mathscr{D}_X$ has a natural filtration (by the order of the operators) and that the associated graded ring is commutative, it is not too difficult to apply the tools of algebraic geometry to this non-commutative setting. In particular, one can define the characteristic variety $\mathrm{char}(\mathscr{M})$ of a coherent $\mathscr{D}_X$-module, a closed $\mathbb{C}^\times$-conic complex analytic subset of the cotangent bundle T^*X and a fundamental result of the theory is that this variety is co-isotropic (or involutive). Partial results in this direction (involutivity at generic points) were first obtained by Guillemin, Quillen, and Sternberg [GQS 70]. The general case was obtained later by Sato, Kawai, and Kashiwara [SKK 73], using tools of microlocal analysis such as microdifferential operators of infinite order. Then Gabber proposed a purely algebraic proof of this result in [Ga 81], and there is also now another totally different proof based on the involutivity of the microsupport of sheaves on a real manifold (see [KS90]).

Once one knows that the characteristic variety of a coherent $\mathscr{D}_X$-module is co-isotropic, it is natural to study with a special attention those modules whose characteristic variety is as small as possible, that is, Lagrangian, and these are the holonomic D-modules. They were first called "maximally overdetermined systems" in [SKK73], and they are the natural generalization in higher dimension of the classical theory of ordinary differential equations. An ordinary differential equation may also be regarded as a connection with poles, and among them, there are the connections with regular singularities or, equivalently, the Fuchsian differential operators. In this framework, the Riemann–Hilbert problem is, roughly speaking, to construct a Fuchsian operator on a Riemann surface when the monodromy of its holomorphic solutions is prescribed.

A natural question is to generalize the theory of Fuchsian equations to higher dimensions. A first important step is the book of Deligne [De70], in which he solves the Riemann–Hilbert problem for regular connections with singularities on hypersurfaces.

A second important step is the constructibility theorem of [Ka75], which asserts that the functor "holomorphic solutions" sends the derived category of holonomic $\mathscr{D}_X$-modules to that of constructible sheaves on X. More precisely, denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ the bounded derived category of left $\mathscr{D}_X$-modules with holonomic cohomology and by $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ the bounded derived category of sheaves of $\mathbb{C}$-vector spaces with constructible cohomologies. Then it is proved in [Ka75] that the (contravariant) functor $\mathcal{S}ol_X(\bullet) = \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\bullet, \mathscr{O}_X)$, when restricted to $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$, takes its values in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. It is also noticed in this paper that if an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ is in the image of the abelian category $\mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$ of holonomic $\mathscr{D}_X$-modules, then it satisfies the properties which are now called the perversity conditions.

It is well known that the functor $\mathcal{S}ol_X \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ is not faithful. For example, if $X = \mathbb{A}^1(\mathbb{C})$, the complex line with coordinate t, $P = t^2\partial_t - 1$ and $Q = t^2\partial_t + t$, then the two holonomic $\mathscr{D}_X$-modules $\mathscr{D}_X/\mathscr{D}_X P$ and $\mathscr{D}_X/\mathscr{D}_X Q$ have the same sheaves of solutions. Hence, a natural question is to look for a full triangulated category of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ on which $\mathcal{S}ol_X$ is fully faithful and induces an equivalence with $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. A precise formulation was formulated in 1977 by the same author (see [Ra78, p. 287]), and a detailed sketch of proof of the theorem establishing this correspondence (in the regular case) appeared in [Ka80] where the functor *Thom* of tempered cohomology was introduced; a detailed proof appears in [Ka84]. Many tools used in the proof of this result were first elaborated in [KK 81]. Note that a quite different proof to this correspondence was obtained by Mebkhout in [Me 84].

The functor *Thom* is thus an essential tool in the original proof of the regular Riemann–Hilbert correspondence. Its functorial properties as well as the construction of the Whitney tensor product $\overset{\mathrm{w}}{\otimes}$, a kind of "dual functor" of *Thom*, are systematically studied in [KS 96]. These two functors are in fact better understood in the language of indsheaves of [KS 01]. They correspond to the indsheaves $\mathscr{O}_X^{\mathrm{t}}$ and $\mathscr{O}_X^{\mathrm{w}}$ of tempered holomorphic functions and Whitney holomorphic functions. For example, $\mathscr{O}_X^{\mathrm{t}}$ is constructed as the Dolbeault complex with tempered distributions as coefficients. Of course, the presheaf of tempered distributions (on a real analytic manifold) is not a sheaf for the usual topology, but it becomes a sheaf for a suitable Grothendieck topology, called the subanalytic topology, and one can naturally embed the category of subanalytic sheaves in that of indsheaves.

Already, in early 2000, it became clear that the indsheaf $\mathscr{O}_X^{\mathrm{t}}$ is an essential tool for the study of irregular holonomic modules. A toy model was studied in [KS 03], where the indsheaf of tempered holomorphic solutions of the ordinary differential operator $t^2\partial_t + 1$ is calculated. However, on $X = \mathbb{A}^1(\mathbb{C})$, the two holonomic $\mathscr{D}_X$-modules $\mathscr{D}_X \exp(1/t)$ and $\mathscr{D}_X \exp(2/t)$ have the same tempered holomorphic solutions, which shows that $\mathscr{O}_X^{\mathrm{t}}$ is not precise enough to treat irregular holonomic D-modules.

This difficulty is overcome in [DK 13] by adding an extra variable in order to capture the growth at singular points. This is done first by adapting to indsheaves a construction of Tamarkin [Ta 08], leading to the notion of "enhanced indsheaves", then by defining the "enhanced indsheaf of tempered holomorphic functions". Using fundamental results of Mochizuki [Mo 09, Mo 11] (see also Sabbah [Sa 00] for preliminary results and see Kedlaya [Ke 10, Ke 11] for the analytic case), this leads to the solution of the Riemann–Hilbert correspondence for (not necessarily regular) holonomic D-modules.

First, we shall recall the main results of the theory of indsheaves and subanalytic sheaves and we shall explain with some detail the operations on D-modules and their tempered holomorphic solutions. As an application, we obtain the Riemann–Hilbert correspondence for regular holonomic D-modules as well as the fact that the de Rham functor commutes with integral transforms.

Second, we do the same for the sheaf of enhanced tempered solutions of (no longer necessarily regular) holonomic D-modules. For that purpose, we first recall the main results of the theory of indsheaves on bordered spaces and its enhanced version.

Let us describe with some details the contents of this book.

Chapter 1 is a review on the theory of sheaves and D-modules. Sheaf theory is now so classical that it does not seem necessary to recall it, and our aim is essentially to establish the notation and to recall the main formulas of constant use. Reference for this subject is made to [KS90]. On the other hand, D-module theory is not so well known. Our presentation of the subject here may be considered as an invitation to the reading of [Ka 03].

In **Chapter 2**, extracted from [KS 96, KS 01], we briefly describe the category of indsheaves on a locally compact space and the six operations on indsheaves. A method for constructing indsheaves on a subanalytic space is the use of the subanalytic Grothendieck topology, a topology for which the open sets are the open relatively compact subanalytic subsets and the coverings are the finite coverings. On a real analytic manifold M, this allows us to construct the indsheaves of Whitney functions, tempered C^∞-functions and tempered distributions. On a complex manifold X, by taking the Dolbeault complexes with such coefficients, we obtain the indsheaf (in the derived sense) $\mathscr{O}_X^{\mathrm{w}}$ of Whitney holomorphic functions and the indsheaf $\mathscr{O}_X^{\mathrm{t}}$ of tempered holomorphic functions.

Then, in **Chapter 3**, also extracted from [KS 96, KS 01], we study the tempered de Rham and Sol ("Sol" for solutions) functors; that is, we study these functors with values in the sheaf of tempered holomorphic functions. We prove two main results which will be the main tools to treat the regular Riemann–Hilbert correspondence later. The first one is Theorem 3.1.1, which calculates the inverse image of the tempered de Rham complex. It is a reformulation of a theorem of [Ka 84], a vast generalization of the famous Grothendieck theorem on the de Rham cohomology of algebraic varieties. The second result, Theorem 3.1.5, is a tempered version of the Grauert direct image theorem.

In **Chapter 4**, we give a proof of the main theorem of [Ka 80, Ka 84] on the Riemann–Hilbert correspondence for regular holonomic D-modules (see Corollary 4.3.4). Our proof is based on Lemma 4.1.9, which essentially claims that to prove that regular holonomic D-modules have a certain property, it is enough to check that this property is stable by projective direct images and is satisfied by modules of "regular normal forms", that is, modules associated with equations of the type $z_i\partial_{z_i} - \lambda_i$ or ∂_{z_j}. The Riemann–Hilbert correspondence as formulated in [Ka80, Ka84] is not enough to treat integral transform, and we have to prove a "tempered" version of it (Theorem 4.3.2). We then collect all results on the tempered solutions of D-modules in a single formula which, roughly speaking, asserts that the tempered de Rham functor commutes with integral transforms whose kernel is regular holonomic (Theorem 4.4.2). We end this chapter with a detailed study of the irregular holonomic D-module $\mathscr{D}_X \cdot \exp(1/z)$ on $\mathbb{A}^1(\mathbb{C})$, following [KS 03]. This case shows that the solution functor with values in the indsheaf $\mathscr{O}_X^{\mathrm{t}}$ gives much information on the holonomic D-modules, but not enough: it is not fully faithful. As seen in the next chapters, in order to treat irregular case, we need the enhanced version of the setting discussed in this chapter.

Chapter 5, extracted from [DK 13], treats indsheaves on bordered spaces. A bordered space is a pair $(M, \widehat{M})$ of good topological spaces with $M \subset \widehat{M}$ an open embedding. The derived category of indsheaves on $(M, \widehat{M})$ is the quotient of the category of indsheaves on $\widehat{M}$ by that of indsheaves on $\widehat{M} \setminus M$. Indeed, contrary to the case of usual sheaves, this quotient is not equivalent to the derived category of indsheaves on M.

The main way of treating the irregular Riemann–Hilbert correspondence is to replace the indsheaf $\mathscr{O}_X^{\mathrm{t}}$ with an enhanced version, the object $\mathscr{O}_X^{\mathsf{E}}$. Roughly speaking, this object (which is no longer an indsheaf) is obtained as the image of the complex of solutions of the operator $\partial_t - 1$ acting on $\mathscr{O}_{X\times\mathbb{C}}^{\mathrm{t}}$, in a suitable category, namely that of enhanced indsheaves.

Chapter 6, also extracted from [DK 13], defines and studies the triangulated category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ of enhanced indsheaves on M, adapting to indsheaves a construction of Tamarkin [Ta 08]. Denoting by $\mathbb{R}_\infty$ the bordered space $(\mathbb{R}, \overline{\mathbb{R}})$ in which $\overline{\mathbb{R}}$ is the two-point compactification of $\mathbb{R}$, the category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ is the quotient of the category of indsheaves on $M \times \mathbb{R}_\infty$ by the subcategory of indsheaves which are isomorphic to the inverse image of indsheaves on M.

Chapter 7, mainly extracted from [DK 13], treats the irregular Riemann–Hilbert correspondence. Similarly as in the regular case, an essential tool is Lemma 7.5.5, which asserts that to prove that holonomic D-modules have a certain property, it is enough to check that this property is stable by projective direct images and is satisfied by modules of "normal forms", that is, D-modules of the type $\mathscr{D}_X \cdot \exp\varphi$ where φ is a meromorphic function. This lemma follows directly from the fundamental results of Mochizuki [Mo 09, Mo 11] (in the algebraic setting) and later Kedlaya [Ke 10, Ke 11] in the analytic case, after preliminary results by Sabbah [Sa 00]. The proof of the irregular Riemann–Hilbert correspondence is rather intricate and uses enhanced constructible sheaves and a duality result between the enhanced solution functor and the enhanced de Rham functor. However, as formulated in [DK 13], this theorem is not enough to treat irregular integral transform and we have to prove an "enhanced" version of it (Theorem 7.8.1, extracted from [KS 14]).

In **Chapter 8**, extracted from [KS 14], we apply the preceding results. The main formula (8.1.4) asserts, roughly speaking, that the enhanced de Rham functor commutes with integral transforms with irregular kernels. In a previous paper [KS 97] we had already proved (without the machinery of enhanced indsheaves) that given a complex vector space $\mathbb{V}$, the Laplace transform induces an isomorphism of the Fourier–Sato transform of the conic sheaf associated with $\mathcal{O}^{\mathsf{t}}_{\mathbb{V}}$ with the similar sheaf on $\mathbb{V}^*$ (up to a shift). We obtain here a similar result in a non-conic setting, replacing $\mathcal{O}^{\mathsf{t}}_{\mathbb{V}}$ with its enhanced version $\mathcal{O}^{\mathsf{E}}_{\mathbb{V}}$. For that purpose, we extend first the Tamarkin non-conic Fourier-Sato transform to the enhanced setting.

Comments. As already mentioned, most of the results discussed here are already known. We sometimes do not give proofs or give only a sketch of the proof. However, Theorems 2.5.13 and 6.6.4 and Corollaries 2.5.15 and 7.7.2, proving the $\mathbb{R}$-constructibility of tempered and Whitney holomorphic solutions of (irregular) holonomic D-modules, are new.

1

A review on sheaves and D-modules

As already mentioned in the Introduction, we assume the reader is familiar with the language of sheaves and D-modules, in the derived sense. Hence, the aim of this chapter is mainly to establish some notation.

1.1 Sheaves

We refer to [KS90] for all notions of sheaf theory used here. For simplicity, we denote by $\mathbf{k}$ a field, although most of the results would remain true when $\mathbf{k}$ is a commutative ring of finite global dimension.

A topological space is *good* if it is Hausdorff, locally compact, countable at infinity and has finite flabby dimension. Let M be such a space. For a subset $A \subset M$, we denote by $\overline{A}$ its closure and $\mathrm{Int}(A)$ its interior.

One denotes by $\mathrm{Mod}(\mathbf{k}_M)$ the abelian category of sheaves of $\mathbf{k}$-modules on M and by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ its bounded derived category. Note that $\mathrm{Mod}(\mathbf{k}_M)$ has a finite homological dimension.

For a locally closed subset A of M, one denotes by $\mathbf{k}_A$ the constant sheaf on A with stalk $\mathbf{k}$ extended by 0 on $X \setminus A$. For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, one sets $F_A := F \otimes \mathbf{k}_A$. One denotes by $\mathrm{Supp}(F)$ the support of F.

We shall make use of the dualizing complex on M, denoted by ω_M, and the duality functors

$$\mathrm{D}'_M := \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathbf{k}_M), \quad \mathrm{D}_M := \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \omega_M). \tag{1.1.1}$$

Recall that, when M is a real manifold, ω_M is isomorphic to the orientation sheaf shifted by the dimension.

We have the two internal operations of internal hom and tensor product:

$$\mathrm{R}\mathscr{H}om\,(\bullet,\bullet) : \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M),$$
$$\bullet \otimes \bullet : \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M).$$

Hence, $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ has a structure of commutative tensor category with $\mathbf{k}_M$ as unit object, and $\mathrm{R}\mathscr{H}om$ is the inner hom of this tensor category.

Now let $f\colon M \to N$ be a morphism of good topological spaces. One has the functors

$$f^{-1}\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \text{ inverse image},$$
$$f^{\,!}\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \text{ extraordinary inverse image},$$
$$\mathrm{R}f_*\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N) \text{ direct image},$$
$$\mathrm{R}f_!\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N) \text{ proper direct image}.$$

We get the pairs of adjoint functors $(f^{-1}, \mathrm{R}f_*)$ and $(\mathrm{R}f_!, f^{\,!})$.

The operations associated with the functors $\otimes, \mathrm{R}\mathscr{H}om, f^{-1}, f^{\,!}, \mathrm{R}f_*$, and $\mathrm{R}f_!$ are called Grothendieck's six operations.

For two topological spaces M and N, one defines the functor of external tensor product

$$\bullet \boxtimes \bullet : \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times N})$$

by setting $F \boxtimes G := q_1^{-1}F \otimes q_2^{-1}G$, where q_1 and q_2 are the projections from $M \times N$ to M and N, respectively.

Denote by pt the topological space with a single element and by $a_M\colon M \to \mathrm{pt}$ the unique morphism. One has the isomorphism

$$\mathbf{k}_M \simeq a_M^{-1}\mathbf{k}_{\mathrm{pt}}, \quad \omega_M \simeq a_M^{!}\mathbf{k}_{\mathrm{pt}}.$$

There are many important formulas relying on the six operations. In particular we have the formulas below in which $F, F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, $G, G_1, G_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$:

$$\mathrm{R}\mathscr{H}om\,(F \otimes F_1, F_2) \simeq \mathrm{R}\mathscr{H}om\,\big(F, \mathrm{R}\mathscr{H}om\,(F_1, F_2)\big),$$
$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(f^{-1}G, F) \simeq \mathrm{R}\mathscr{H}om\,(G, \mathrm{R}f_*F),$$
$$\mathrm{R}f_!(F \otimes f^{-1}G) \simeq (\mathrm{R}f_!F) \otimes G \qquad \text{(projection formula)},$$
$$f^{\,!}\mathrm{R}\mathscr{H}om\,(G_1, G_2) \simeq \mathrm{R}\mathscr{H}om\,(f^{-1}G_1, f^{\,!}G_2),$$

and for a Cartesian square of good topological spaces,

$$\begin{tikzcd} M' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & N' \arrow[d, "g"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{1.1.2}$$

we have the *base change formulas for sheaves*

$$g^{-1}\mathrm{R}f_! \simeq \mathrm{R}f'_! g'^{-1} \quad \text{and} \quad g^!\mathrm{R}f_* \simeq \mathrm{R}f'_* g'^!.$$

In this book, we shall also encounter $\mathbb{R}$-*constructible* sheaves. References are made to [KS90, ch. 8]. Let M be a real analytic manifold. On M there is the family of subanalytic sets due to Hironaka and Gabrielov (see [BM88, VD98] for an exposition). This family is stable by all usual operations (finite intersection and locally finite union, complement, closure, interior) and contains the family of semi-analytic sets (those locally defined by analytic inequalities). If $f: M \to N$ is a morphism of real analytic manifolds, then the inverse image of a subanalytic set is subanalytic. If Z is subanalytic in M and f is proper on the closure of Z, then $f(Z)$ is subanalytic in N.

A sheaf F is $\mathbb{R}$-*constructible* if there exists a subanalytic stratification $M = \bigsqcup_{j\in J} M_j$ such that for each $j \in J$, the sheaf $F|_{M_j}$ is locally constant of finite rank. One defines the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ as the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of objects F such that $H^i(F)$ is $\mathbb{R}$-constructible for all $i \in \mathbb{Z}$ and one proves that this category is triangulated.

The category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ is stable by the usual internal operations (tensor product, internal hom), and the duality functors in (1.1.1) induce anti-equivalences on this category.

If $f: M \to N$ is a morphism of real analytic manifolds, then f^{-1} and $f^!$ send $\mathbb{R}$-constructible objects to $\mathbb{R}$-constructible objects. If $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ and f is proper on $\mathrm{Supp}(F)$, then $\mathrm{R}f_!F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_N)$.

1.2 D-modules

References for D-module theory are made to [Ka03]. See also [Ka70, Bj93, HTT08].

Here, we shall briefly recall some basic constructions in the theory of D-modules that we shall use. Note that many classical functors that shall appear in this chapter will be extended to indsheaves in Chapter 3 and the subsequent chapters.

In this section, the base field is the complex number field $\mathbb{C}$.

1.2.1 Basic constructions

Let $(X, \mathscr{O}_X)$ be a *complex* manifold. We denote as usual by

- d_X the complex dimension of X,
- Ω_X the invertible sheaf of differential forms of top degree,
- $\Omega_{X/Y}$ the invertible $\mathscr{O}_X$-module $\Omega_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(\Omega_Y^{\otimes -1})$ for a morphism $f: X \to Y$ of complex manifolds,
- Θ_X the sheaf of holomorphic vector fields,
- $\mathscr{D}_X$ the sheaf of algebras of finite-order differential operators, the subring of $\mathscr{H}om\,(\mathscr{O}_X, \mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X.

Denote by $\mathrm{Mod}(\mathscr{D}_X)$ the abelian category of left $\mathscr{D}_X$-modules and by $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}})$ that of right $\mathscr{D}_X$-modules. There is an equivalence

$$\mathrm{r}\colon \mathrm{Mod}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}), \quad \mathscr{M} \mapsto \mathscr{M}^{\mathrm{r}} := \Omega_X \otimes_{\mathscr{O}_X} \mathscr{M}. \tag{1.2.1}$$

By this equivalence, it is enough to study left $\mathscr{D}_X$-modules.

1.2.2 Filtrations and characteristic variety

The ring $\mathscr{D}_X$ is endowed with the filtration by the order. Denoting by $\mathrm{F}\,\mathscr{D}_X$ this filtered ring, $\mathrm{F}_m\,\mathscr{D}_X$ is the sheaf of differential operators of order $\le m$. One can also define this filtration by

$$\mathrm{F}_{-1}\,\mathscr{D}_X = \{0\}, \quad \mathrm{F}_m\,\mathscr{D}_X = \{P \in \mathscr{D}_X; [P, \mathscr{O}_X] \in \mathrm{F}_{m-1}\,\mathscr{D}_X\}.$$

Note that

$$\begin{cases} \mathrm{F}_0\,\mathscr{D}_X = \mathscr{O}_X, \quad \mathrm{F}_1\,\mathscr{D}_X = \mathscr{O}_X \oplus \Theta_X, \\ \mathrm{F}_m\,\mathscr{D}_X \cdot \mathrm{F}_l\,\mathscr{D}_X \subset \mathrm{F}_{m+l}\,\mathscr{D}_X, \quad [\mathrm{F}_m\,\mathscr{D}_X, \mathrm{F}_l\,\mathscr{D}_X] \subset \mathrm{F}_{m+l-1}\,\mathscr{D}_X. \end{cases} \tag{1.2.2}$$

We denote by $\mathrm{gr}\,\mathscr{D}_X$ the associated graded ring:

$$\mathrm{gr}\,\mathscr{D}_X = \bigoplus_i \mathrm{F}_i\,\mathscr{D}_X / \mathrm{F}_{i-1}\,\mathscr{D}_X,$$

by $\sigma\colon\ \mathrm{F}\,\mathscr{D}_X \to \mathrm{gr}\,\mathscr{D}_X$ the "principal symbol map" and by $\sigma_m : \mathrm{F}_m\,\mathscr{D}_X \to \mathrm{gr}_m\mathscr{D}_X$ the map "symbol of order m."

The ring $\mathrm{gr}\,\mathscr{D}_X$ is a *commutative* graded ring. Moreover, $\mathrm{gr}_0\mathscr{D}_X \simeq \mathscr{O}_X$ and $\mathrm{gr}_1\mathscr{D}_X \simeq \Theta_X$.

Denote by $S_{\mathscr{O}_X}(\Theta_X)$ the symmetric $\mathscr{O}_X$-algebra associated with the locally free $\mathscr{O}_X$-module Θ_X. By the universal property of symmetric algebras, the morphism $\Theta_X \to \mathrm{gr}\,\mathscr{D}_X$ may be extended to a morphism of symmetric algebras

$$S_{\mathscr{O}_X}(\Theta_X) \to \mathrm{gr}\,\mathscr{D}_X, \tag{1.2.3}$$

and one easily proves that the morphism (1.2.3) is an isomorphism.