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Excerpt
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THE STRAIGHT LINE

CHAPTER I

INTRODUCTION

Before commencing a rider read it through very carefully. It is a good plan to underline words or phrases that give rise to definite ideas, such as parallels, the angle BAC is bisected, $AB = AC$, two right angles..., and also what is required to be proved.

All the given facts must be used; the first thing to be done is to associate with each of these facts the properties connected with it. At the head of each chapter in what follows will be found the main properties discussed in that chapter.

The solution of riders consists in the main of eliminating the non-useful properties associated with each fact, and of linking up the remainder. Indications as to the proper sequence of the relevant facts invariably present themselves.

Special properties of any figures mentioned should be noted. Thus: a rectangle is a parallelogram with one of its angles a right angle; the diagonals of a rhombus bisect one another at right angles. These things often furnish a clue to the solution.

Some riders are with advantage worked "from both ends". In other words, some of the steps previous to that giving a final solution are retraced with a view to seeing whether they accord with the conclusions arrived at directly from the given facts. If that is so, the whole solution may then be built up. Examples of this method will be found on pp. 74, 105.

Great care must be taken in the drawing of diagrams. When given a triangle (without any other qualifications), do not draw an isosceles or an equilateral triangle, and be careful not to draw a parallelogram or a rectangle when you are simply given a quadrilateral. It is equally important that, when a figure is given any special characteristics, due attention is paid to them in drawing it. Thus if in a triangle $AB > AC$, or $AB = AC$, do not draw a triangle in which $AB < AC$, or even one in which AB is only slightly greater than AC. A warning, too, must be given against such practices as regarding things as equal because they look equal; assuming two straight lines perpendicular when they seem to cut at right angles. Of course many equalities are often suggested by the appearance of the figure, but such things must not be taken for granted unless they can be proved, and are proved. This does not mean that very accurate figures need be drawn with compasses and set-squares, etc.; a little care will always provide a satisfactory figure without the waste of any time.

It is a good plan to indicate equal lengths or angles in a figure by suitable marking: ⧺, ⧺ for lines, ∠o∠o, ∠×∠× or letters $\alpha\alpha$, $\beta\beta$ for angles, with a small c for common elements.

Proofs should be set out neatly; there should be only one step, however short it may be, to a line. References should be given wherever possible; these should not be made by numbers, but as far as possible by names, e.g.

Three sides (in proving Congruence);
Intercept theorem;
Pythagoras' theorem;
Rectangle theorem;
Vertically opposite;
Angle-sum;...

Finally, it is useful to remember that some riders (and also some constructions) have what may be called a family connection, and are conveniently grouped as such. We have, to give two instances:

The Kites,..., see pp. 12, 80 (q. 9), 107 (q. 11).

Riders in which we have to produce a median to twice its length,..., see pp. 14, 48.

CHAPTER II

PARALLELS

When we think of parallels only three main ideas need be exploited. If they are met by a transversal,

(i) *the alternate angles are equal;*

(ii) *the corresponding angles are equal;*

(iii) *the allied* angles are supplementary.*

There are three other possible ideas, but they are dependent upon particular conditions and can at once be isolated:

(iv) *parallels in connection with areas*—the idea of areas must necessarily be present. We deal with this in chapter VIII;

(v) *parallels in connection with mid-points of the sides of a triangle or trapezium*—for this see chapter V;

(vi) *parallels in connection with the ratio propositions*—this group is at once recognised by a direct reference to, or implication of, ratios.

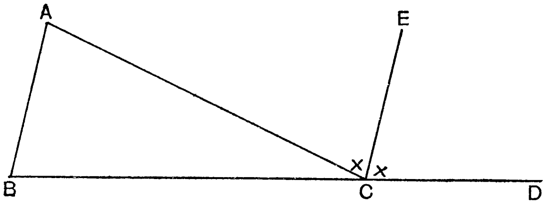
If in a rider, when ideas (iv)–(vi) are not present, parallel straight lines are mentioned, one of the facts (i)–(iii) will have to be used, either in the direct or converse form.

The proof of the properties of a parallelogram is a typical example merely of writing down the consequences of the fact that the opposite sides of a parallelogram are parallel.

* We follow a recently suggested contraction for "interior angles on the same side of the cutting line".

Illustrative Riders

If the external bisector of an angle of a triangle is parallel to the opposite side, the triangle is isosceles.



Ideas:

(i) Mark the equal angles ACE, ECD.

(ii) Keep in mind that an isosceles triangle in terms of angles (for parallels lead us to angles) means two equal angles.

(iii) AB and EC are parallel. We must use this.

We have $B\hat{A}C = A\hat{C}E$ (alternate),

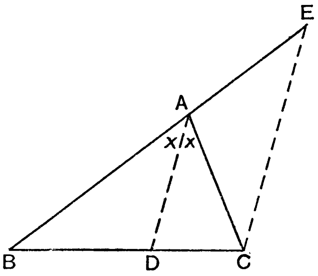
$A\hat{B}C = E\hat{C}D$ (corresponding).

[Note that we have brought in the angles ACE, ECD about which we know something, namely that they are equal.]

So $B\hat{A}C = A\hat{B}C$,

i.e. $AC = BC$.

AD, the internal bisector of the $B\hat{A}C$ *of a* $\triangle ABC$, *meets BC at D; a parallel through C to AD meets BA produced at E. Prove that* $AC = AE$.



Ideas:

(i) We have to prove

$$AC = AE.$$

Translating this into terms of angles (because of parallels), we see that we have to prove

$$A\hat{C}E = A\hat{E}C.$$

(ii) What use can we make of the fact that AD is parallel to CE? Keep $A\hat{C}E$ and $A\hat{E}C$ in mind.

We have $A\hat{C}E = D\hat{A}C$ (alternate),

$A\hat{E}C = B\hat{A}D$ (corresponding).

This must help us, as we are bringing $B\hat{A}D$ and $D\hat{A}C$ into the question.

(iii) Use $B\hat{A}D = C\hat{A}D$

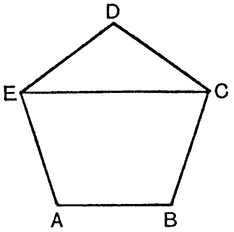
and we have at once

$$A\hat{C}E = A\hat{E}C.$$

The solution may now be reconstructed.

Riders into which "allied" angles enter have a direct connection with one or two right angles, and the angle-sum theorem will frequently be necessary. The following is a typical example:

ABCDE is a regular pentagon. Prove that CE is parallel to AB.



(We shall here assume the properties of a regular pentagon, i.e.

$$AB = BC = CD = DE = EA$$

and $\hat{A} = \hat{B} = \text{etc.} = 108°$;

and the angle-sum theorem see p. 39.)

Ideas:

(i) To prove CE parallel to AB we must have:

(*a*) a pair of alternate angles equal;

or (*b*) a pair of corresponding angles equal;

or (*c*) a pair of allied angles supplementary.

(ii) What do we know?

We have $E\hat{A}B = 108°$. Can we evaluate $A\hat{E}C$ and use (*c*)? (To make use of (*a*) or (*b*) we should have to produce CE or AE, and it would still be necessary to evaluate $A\hat{E}C$.)

(iii) Of the angles at E we know $A\hat{E}D = 108°$. Can we evaluate $D\hat{E}C$?

Now $D\hat{E}C$ is an angle of a triangle—an isosceles triangle.

But $E\hat{D}C = 108°$,

so $D\hat{E}C = D\hat{C}E = \tfrac{1}{2}.(180 - 108°)$

$= 36°$.

$\therefore A\hat{E}C = 108° - 36°$

$= 72°$,

so $E\hat{A}B + A\hat{E}C = 108° + 72°$

$= 180°$.

$\therefore$ AB and CE are parallel.

Further examples having a bearing on this chapter are worked on pp. 37, 96 and 102.

Exercise 1

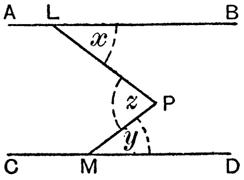
1. Prove that straight lines which are parallel to the same straight line are parallel to one another.

2. If a straight line is perpendicular to one of a system of parallel straight lines, it is perpendicular to all the lines of the system.

3. If the opposite sides of a quadrilateral are parallel, its opposite angles are equal.

4. Similarly for a hexagon, or for any figure with an even number of sides.

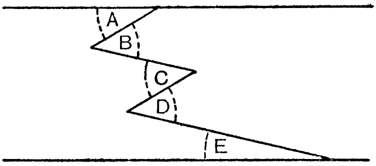
5. AB and CD are two parallel straight lines; L is any point in AB, M is any point in CD. Prove (see figure) that $z = x + y$.



6. Prove, similarly, in this figure that

$$B + D = A + C + E.$$

Give a generalised form of this rider.



7. If two intersecting straight lines are respectively parallel to two other intersecting straight lines, prove that the angle between the first pair is equal, or supplementary, to that between the second pair.

8. Prove that in an isosceles triangle the external bisector of the vertical angle is parallel to the base.

9. In a $\triangle ABC$, $AB = AC$. DE is drawn parallel to BC to meet AB and AC respectively at D and E. Prove that $AD = AE$.

10. Straight lines are drawn through the vertices of a triangle parallel to the opposite sides. Prove that the triangle so formed is similar to the original triangle (i.e. that they are equiangular).

11. AD is the external bisector of the angle BAC of a $\triangle ABC$. CE is drawn parallel to AD to meet AB at E. Prove that $AC = AE$.

[You will have less difficulty with your figure if you make angle ACB obtuse.]

12. Two straight lines AB, CD intersect at O. If the bisectors of the adjacent angles AOC, COB meet a parallel through C to AB in P and Q, prove that $PC = OC = CQ$.

13. The internal bisectors of the angles B and C of a $\triangle ABC$ meet at O. A straight line is drawn through O parallel to BC and meets AB, AC at D and E respectively. Prove that

$$DE = BD + CE.$$

14. The equal angles B and C of an isosceles triangle ABC are bisected by lines BY, CX which meet the opposite sides in Y and X. Prove that XY is parallel to BC and also that

$$BX = XY = YC.$$

CHAPTER III

CONGRUENCE

To prove the congruence of two triangles it is necessary that the following elements of the one triangle be equal to the corresponding elements of the other:

(i) *two sides and the angle included by them;*

or (ii) *the three sides;*

or (iii) *two angles and a side.*

There is one special case:

(iv) *if the two triangles are right angled, it is sufficient that any two sides of the one are equal to the corresponding sides of the other.*

The procedure to be followed in the proving of congruence is as follows. Examine the two triangles to see if the conditions of any of the above cases hold. If so, the proof may at once be constructed. Notice particularly whether the triangles have a common side or a common angle, or whether vertically opposite angles appear.

Frequently only two equal elements are given. If *two angles* of one triangle are equal to the corresponding angles of the other triangle, look out for a pair of equal sides (or a common side), or for a means of proving a pair of sides equal (e.g. the isosceles triangle theorem).

If *two sides* of one triangle are equal to the corresponding sides of the other, examine the figure to see if it is at all possible to prove their included angles equal. If not, try the third sides.

If *one side and one angle* of a triangle are equal to the corresponding side and angle of the other then, if the side and the angle face each other and the equal angles are acute, congruence can only be proved by first establishing the equality of a second