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Alan F. Beardon
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# 1

# Paying for parking

## 1.1 The problem

Suppose that you have to pay for parking your car by putting coins in a machine that only accepts coins of values $p$ and $q$ units. *Which charges can you pay for without requiring any change*? The answer is obvious if $p = 1$, or $q = 1$, or $p = q$, so, from now on, *we shall assume that* $p \geqslant 2$, $q \geqslant 2$ and $p \neq q$. Clearly, the set of charges that you can pay for without requiring change is

$$M(p, q) = \{mp + nq : m, n = 0, 1, \dots\},$$

and the problem is to say as much as you can about this set. From now on we shall omit the phrase 'without requiring change', although we shall always assume that this condition applies.

To begin, for a given pair $(p, q)$, say $(3, 5)$, you might mark the points $(mp, nq)$, $m, n = 0, 1, 2, \dots$ on graph paper and see whether any ideas emerge from this picture.

## 1.2 Coprime values

It is obvious that if $p$ and $q$ are even then we can only pay an even number of units. More generally, if $k$ is the greatest common divisor of $p$ and $q$, that is, $\gcd(p, q) = k$, then we can only pay amounts that are integer multiples of $k$. These comments suggest that we proceed as follows. Let $d$ be the greatest common divisor of $p$ and $q$, and let $p_1 = p/d$ and $q_1 = q/d$. Then $p_1$ and $q_1$ are coprime integers, and (in the obvious sense)

$$M(p,q) = d\,\{mp_1 + nq_1 : m, n = 0, 1, \ldots\} = d\,M(p_1, q_1).$$

This argument shows that it is sufficient to consider $M(p_1, q_1)$ or, equivalently, to restrict ourselves to the case where $p$ and $q$ are coprime. Thus, from now on, *we shall assume that $p$ and $q$ are coprime*; that is, $\gcd(p,q) = 1$. How does this help? Well, it is extra information which we should be able to use to make further progress, *but only if we know some facts about coprime integers*. Thus we must now turn to number theory.

The most important consequence of $\gcd(p,q) = 1$ is that *there are integers $r$ and $s$ such that $rp + sq = 1$*. Briefly, we recall the proof. Consider the group $G = \{mp + nq : m, n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers. As $\mathbb{Z}$ is a cyclic group, and $G$ is a subgroup of $\mathbb{Z}$, we see that $G$ is cyclic. Thus we can write $G = \{kg : k \in \mathbb{Z}\}$, where $g > 0$, so that

$$\{mp + nq : m, n \in \mathbb{Z}\} = G = \{kg : k \in \mathbb{Z}\}.$$

It follows that there are integers $k_1$ and $k_2$ with $p = 1.p + 0.q = k_1 g$ and $q = 0.p + 1.q = k_2 g$. We deduce that $g$ divides both $p$ and $q$, so that $g = 1$, and then $\{mp + nq : m, n \in \mathbb{Z}\} = G = \mathbb{Z}$. As $1 \in \mathbb{Z}$, this shows that $mp + nq = 1$ for some $m$ and $n$.

## 1.3 Which charges require change?

The reader should now carry out a few numerical experiments (on a computer), and these should suggest the following preliminary result.

**Lemma 1.1** *Suppose that* $\gcd(p,q) = 1$. *Then there are only a finite number of charges which cannot be paid.*

Note that as the conclusion of Lemma 1.1 is false when $p$ and $q$ are not coprime, we will have to use the assumption that $\gcd(p,q) = 1$ somewhere in our proof. We shall illustrate the idea behind a general proof with a specific example, namely when $p = 5$ and $q = 11$. The first observation is that if we can pay five *consecutive* charges, say $N, N+1, \ldots, N+4$, then we can pay all amounts above $N$ (simply by using more coins of value 5). Thus the problem is reduced to showing that we can pay five consecutive amounts.

Now as $\gcd(5, 11) = 1$, we can solve $5m + 11n = 1$, say with $m = -2$ and $n = 1$ (there are other solutions here, for example, $m = 9$ and $n = -4$). This means that to pay an additional unit charge, we can put *one more* 11-*unit coin* in the machine and *two fewer* 5-*unit coins*. Of course, this is possible only if we have already put two (or more) 5-unit coins in the machine. Suppose that we can pay an amount $N$ with, say $a$ coins of 5 units and $b$ coins of 11 units. Then $N = 5a + 11b$ and

$$\begin{aligned}
N+1 &= 5a + 11b + \big(1.11 + (-2)5\big) = (a-2)5 + (b+1)11,\\
N+2 &= 5a + 11b + \big(2.11 + (-4)5\big) = (a-4)5 + (b+2)11,\\
N+3 &= 5a + 11b + \big(3.11 + (-6)5\big) = (a-6)5 + (b+3)11,\\
N+4 &= 5a + 11b + \big(4.11 + (-8)5\big) = (a-8)5 + (b+4)11.
\end{aligned}$$

Clearly, all these payments are possible if $a \geqslant 8$ and $b \geqslant 0$, so, for example, by taking $a = 8$ and $b = 0$ we see that it is possible to pay each of the amounts $40, 41, \ldots, 44$.

**Problem 1.1** Generalise the argument given above to provide a proof of Lemma 1.1. The starting data is a pair of coprime, positive integers $p$ and $q$, and the existence of integers $u$ and $v$ such that $pu + qv = 1$, where (necessarily) either $u < 0 < v$ or $v < 0 < u$. Now show that, providing $N$ is sufficiently large, it is possible to pay each of the charges $N, N+1, \ldots, N+p$. How large must $N$ be (in terms of $p$ and $q$) for this to be possible?

As is so often the case, a positive result raises further questions.

**Problem 1.2** What is the largest amount that *cannot* be paid? How many different values *cannot* be paid?

Notice how our interest has changed from amounts that *can* be paid to amounts that *cannot* be paid. A change of emphasis is often crucial in solving problems.

**Problem 1.3** The reader should now use a computer and show experimentally that the following result is true.

**Theorem 1.2** *Suppose that* $\gcd(p, q) = 1$. *Then we cannot pay a charge of* $pq - p - q$, *but all charges above this amount can be paid.*

*Proof* First, we show that we cannot pay $pq-p-q$, and to do this we argue by contradiction. We suppose, then, that we can pay this amount; thus we assume that there are non-negative integers $u$ and $v$ such that $pq-p-q=up+vq$, or

$$pq=(u+1)p+(v+1)q.$$

Since $\gcd(p,q)=1$, this shows that $p$ divides $v+1$, so that $v+1\geqslant p$. Likewise, $u+1\geqslant q$. Thus $pq=(u+1)p+(v+1)q\geqslant 2pq$, which is false. This contradiction shows that we cannot pay $pq-p-q$ units.

In order to show that all charges greater than $pq-p-q$ can be paid, we choose any integer $n$ with $n>pq-p-q$. Next, as $\gcd(p,q)=1$, there are integers $a$ and $b$ such that $pa+bq=1$; thus $(na)p+(nb)q=n$. Now we can always write $na$ as a multiple of $q$ plus a remainder, say $na=cq+d$, where $0\leqslant d<q$. Then

$$\begin{aligned} pq-p-q<n &= (na)p+(nb)q \\ &= pd+(pc+nb)q \\ &\leqslant p(q-1)+(pc+nb)q. \end{aligned}$$

This shows that $pc+nb>-1$ so that $pc+nb\geqslant 0$. Since $n=pd+(pc+nb)q$, it follows that a charge of $n$ can be paid (with $d$ coins of value $p$ and $pc+nb$ coins of value $q$). □

## 1.4 Which charges cannot be paid?

We now ask *how many* charges cannot be paid?

**Problem 1.4** The reader should provide numerical examples to show *experimentally* that exactly $\frac{1}{2}(p-1)(q-1)$ charges cannot be paid. Note that as $\gcd(p,q)=1$, one of $p$ and $q$ must be odd, so that $(p-1)(q-1)$ is always an even integer.

**Theorem 1.3** *Suppose that* $\gcd(p,q)=1$. *Then* $\frac{1}{2}(p-1)(q-1)$ *charges cannot be paid.*

*Proof* We begin by considering the number $N(k)$ of ways that we can pay for a charge $k$. Clearly, $N(k)$ is the number of pairs $(u,v)$ of non-negative integers $u$ and $v$ with $pu+qv=k$. As there are integers $a$ and

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$b$ with $pa+qb=1$, there are certainly integers $u$ $(=ka)$ and $v$ $(=kb)$ such that $pu+qv=k$. It is well known that this means that the general solution in integers of $px+qy=k$ is given by $(x,y)=(u+qt, v-pt)$, where $t\in\mathbb{Z}$. These points lie on the line $L_k$ given by $px+qy=k$ in $\mathbb{R}^2$ and are equally spaced along $L_k$, with two consecutive points being a distance $\sqrt{p^2+q^2}$ apart (the reader should draw a diagram here). Now the line $L_k$ meets the region $\{(x,y) : x\geqslant 0, y\geqslant 0\}$ in the segment, say $S_k$, with endpoints $(k/p,0)$ and $(0,k/q)$, and $S_k$ has length $(k/pq)\sqrt{p^2+q^2}$. It follows that if $k<pq$ then there is at most one point $(x,y)$ with integer co-ordinates on $S_k$; hence *at most one way that we can pay a charge* $k$. We leave the reader to show that there are exactly two ways to pay a charge of $pq$ (namely $q$ coins of value $p$, or $p$ coins of value $q$), and these correspond to the points $(q,0)$ and $(0,p)$ which are, in fact, the endpoints of the seqment $S_{pq}$.

A *lattice point* is a point in $\mathbb{R}^2$ that has integer co-ordinates, and the problem has now been reduced to counting lattice points. Let $T$ be the triangular region given by $x\geqslant 0$, $y\geqslant 0$ and $px+qy<pq$, and let $|T|$ denote the number of lattice points in $T$. We have just seen that there are exactly $|T|$ values of $k$ in $\{0,1,2,\dots,pq-1\}$ that can be paid; hence (since all values above $pq-1$ can be paid), *there are exactly* $pq-|T|$ *non-negative values* $k$ *which cannot be paid*. It remains to find $|T|$.

Now consider the triangular region $T^*$ given by the inequalities $x\leqslant q$, $y\leqslant p$ and $px+qy>pq$ (the reader is advised to draw a diagram). Then the union of the three mutually disjoint sets $T$, $S_{pq}$ and $T^*$ is the rectangle $R$ given by $0\leqslant x\leqslant q$ and $0\leqslant y\leqslant p$. If we now denote the number of lattice points in a set $E$ by $|E|$, we have

$$|T|+2+|T^*|=|T|+|S_{pq}|+|T^*|=|R|=(p+1)(q+1).$$

The last step in the argument is to show that $|T|=|T^*|$; this can be done by considering the map $(x,y)\mapsto(q-x,p-y)$, and we leave this for the reader to complete. With this we have

$$2|T|+2=(p+1)(q+1),$$

which implies that $pq-|T|=\frac{1}{2}(p-1)(q-1)$ as required. □

**Problem 1.5** Given $p$ and $q$, is it possible to say anything about the actual amounts that cannot be paid?

## 1.5 Three denominations of coins

We now ask about the situation when we are allowed to pay using coins of *three* values $p$, $q$ and $r$. Since we can pay any charge that exceeds $(p-1)(q-1)$ by using only coins of values $p$ and $q$, and similarly for $(p, r)$ and $(q, r)$, it is clear that we can pay any charge that exceeds

$$\min\{(p-1)(q-1), (q-1)(r-1), (r-1)(p-1)\}.$$

However, unlike the case of two values, *there is no known explicit expression for the maximum charge that cannot be paid*. It is known that this maximum value is *not* a polynomial in $p$, $q$ and $r$ (see [9]), so this may well be a very difficult problem.

Finally, we can solve specific problems of this type by using computer software that can multiply polynomials. Consider, for example, the problem of paying a charge $N$ with coins of values $p$, $q$ and $r$. Obviously, we need at most $N$ coins of each type, so let

$$\left(\sum_{i=0}^{N} x^{pi}\right)\left(\sum_{j=0}^{N} x^{qj}\right)\left(\sum_{k=0}^{N} x^{rk}\right) = 1 + c_1 x + c_2 x^2 + \cdots .$$

As $c_N$ is the number of terms in the expanded product that are of the form $x^{pi+qj+rk}$, where $i$, $j$ and $k$ are non-negative integers such that $pi + qj + rk = N$, we see that there are exactly $c_N$ ways that we can pay a charge $N$.