CHAPTER I

THEOREMS IN ALGEBRA

1. Before entering upon the subject of elliptic and hyperelliptic integrals, which forms the larger portion of the matter discussed in our Studies, we shall explain some terms and lay before the reader the proofs of some algebraic theorems which we shall frequently find it convenient to employ.

We shall denote the polynomial

$$z^m + p_1 z^{m-1} + p_2 z^{m-2} + \ldots + p_m \text{ by } \phi(z),$$

and we shall call the roots of the equation

$$\phi(z) = 0,\ z_1,\ z_2,\ z_3, \ldots z_m;$$

so that $$\phi(z) \equiv (z - z_1)(z - z_2)(z - z_3) \ldots (z - z_m).$$

We shall refer to these roots as the z's and to the quantities $p_1, p_2, p_3, \ldots p_m$ as the p's.

By the theory of equations we know that

$$\Sigma z_1 = -p_1;\ \Sigma z_1 z_2 = p_2;\ \Sigma z_1 z_2 z_3 = -p_3;\ \Sigma z_1 z_2 z_3 \ldots z_r = (-1)^r p_r;\ \&c.;$$
$$z_1 z_2 z_3 \ldots z_m = (-1)^m p_m;$$

Σz_1 denoting the sum of the z's, $\Sigma z_1 z_2$ the sum of their products taken two together, $\Sigma z_1 z_2 z_3$ the sum of their products taken three together, $\Sigma z_1 z_2 z_3 \ldots z_r$ the sum of their products taken r together.

Now, suppose we divide both sides of the identity

$$(z - z_1)(z - z_2)(z - z_3) \ldots (z - z_m) \equiv z^m + p_1 z^{m-1} + p^2 z^{m-2} + \ldots + p_m$$

by $z - z_1$, we find easily

$$\begin{aligned}(z - z_2)(z - z_3) \ldots (z - z_m)\\ \equiv z^{m-1} + (p_1 + z_1) z^{m-2} + (p_2 + p_1 z_1 + z_1^2) z^{m-3}\\ + (p_3 + p_2 z_1 + p_1 z_1^2 + z_1^3) z^{m-4}\\ + \ldots + (p_{m-1} + p_{m-2} z_1 + p_{m-3} z_1^2 + \ldots + z_1^{m-1}).\end{aligned}$$

Hence, connecting the $m - 1$ quantities $z_2, z_3, \ldots z_m$, we have the following equations:

$$\Sigma z_2 = -p_{1,1} = -(p_1 + z_1);\ \Sigma z_2 z_3 = p_{2,1} = p_2 + p_1 z_1 + z_1^2;$$
$$\Sigma z_2 z_3 z_4 = p_{3,1} = -p_3 + p_2 z_1 + p_1 z_1^2 + z_1^3;\ \&c. \ldots (1),$$

where $z_2, z_3, \ldots z_m$ are the roots of the equation

$$z^{m-1} + p_{1,1} z^{m-2} + p_{2,1} z^{m-3} + \ldots + p_{m-1,1} = 0 \ldots\ldots\ldots (2),$$

and in future we shall adopt this notation as a convenient one, namely that $p_{1,r}, p_{2,r}, p_{3,r}$, &c. will refer to the $m-1$ quantities which remain when we omit z_r from the m z's.

Again, the above notation readily explains itself, so that in all cases of any sum in which the first root z of the typical term is affected with the suffix 2, that sum is to be considered as referring to the $m-1$ quantities $z_2, z_3, \ldots z_m$, and further, if the first root z of the typical term is affected with the suffix 3, that sum is to be considered as referring to the $m-2$ quantities $z_3, z_4, \ldots z_m$ which remain on leaving out z_1 and z_2.

For example, $\Sigma z_2^2 z_3$ refers to the $m-1$ quantities $z_2, z_3, \ldots z_m$, while $\Sigma z_3^2 z_4$, to the $m-2$ quantities $z_3, z_4, \ldots z_m$. For the sake of convenience we shall sometimes omit the suffix, in which case Σ applies to *all* the quantities considered; for instance,

$$\Sigma z^2 \equiv z_1^2 + z_2^2 + z_3^2 + \ldots + z_m^2,$$

$$\Sigma \int \frac{z^i dz}{\sqrt{f(z)}} \equiv \int \frac{z_1^i dz_1}{\sqrt{f(z_1)}} + \int \frac{z_2^i dz_2}{\sqrt{f(z_2)}} + \ldots + \int \frac{z_m^i dz_m}{\sqrt{f(z_m)}}.$$

We have also the following system of equations, which are easily obtained by multiplying

$$z^{m-1} + p_{1,1} z^{m-2} + p_{2,1} z^{m-3} + \ldots + p_{m-1,1} \text{ by } z - z_1,$$

and identifying the result with

$$z^m + p_1 z^{m-1} + p_2 z^{m-2} + \ldots + p_m,$$

$$\left.\begin{array}{l} p_1 = p_{1,1} - z_1;\ \ p_2 = p_{2,1} - z_1 p_{1,1};\ \ p_3 = p_{3,1} - z_1 p_{2,1} \\ \text{\&c.} \end{array}\right\} \ldots (3).$$

2. If we now write

$$\chi(z) \equiv c_1 z^{m-1} + c_2 z^{m-2} + c_3 z^{m-3} + \ldots + c_m,$$

and if $\chi(z)$ vanishes for each of the m quantities $z_1, z_2, z_3, \ldots z_m$, so that

$$\chi(z_1) = 0;\ \ \chi(z_2) = 0;\ \ \chi(z_3) = 0;\ \ \text{\&c};\ \ \chi(z_m) = 0;$$

we say that

$$c_1 = 0;\ \ c_2 = 0;\ \ c_3 = 0;\ \ \text{\&c.};\ \ c_m = 0;$$

it being understood that the equation $\phi(z) = 0$ has no equal roots.

For, since $\chi(z)$ vanishes for each of the $m-1$ quantities

$$z_2,\ z_3,\ z_4,\ \ldots\ z_m,$$

it follows that

$$\chi(z) \equiv c_1 (z - z_2)(z - z_3)(z - z_4) \ldots (z - z_m),$$

and hence by identifying this latter form of $\chi(z)$ with that written above, we find

$$c_2 = -c_1 \Sigma z_2; \quad c_3 = c_1 \Sigma z_2 z_3; \quad c_4 = -c_1 \Sigma z_2 z_3 z_4; \quad \&c.$$

But, by hypothesis $\chi(z_1)$ also vanishes and consequently we must have

$$0 = c_1 (z_1 - z_2)(z_1 - z_3)(z_1 - z_4) \dots (z_1 - z_m),$$

and therefore $c_1 = 0$, and since c_1 vanishes it follows that $c_2, c_3, \dots c_m$ vanish also.

Suppose now that $\chi(z)$ was of the rth degree in z, r being greater than $m-1$, and vanished for each root of $\phi(z) = 0$, χ being defined in the following manner

$$\chi(z) \equiv D_0 z^r + D_1 z^{r-1} + D_2 z^{r-2} + \dots + D_r.$$

We can reduce this case to that just considered as we now proceed to show.

For, if z be a root of $\phi(z) = 0$, we have

$$z^m = -(p_1 z^{m-1} + p_2 z^{m-2} + p_3 z^{m-3} + \dots + p_m),$$

thus expressing z^m in terms of the p's and lower powers of z; again, multiplying the above equation by z, we find

$$z^{m+1} = -p_1 z^m - p_2 z^{m-1} - p_3 z^{m-2} - \dots - p_m z,$$

and, introducing into the right-hand side the value of z^m given above, we have

$$\begin{aligned} z^{m+1} &= p_1 \{p_1 z^{m-1} + p_2 z^{m-2} + p_3 z^{m-3} + \dots + p_m\} \\ &\quad - \{p_2 z^{m-1} + p_3 z^{m-2} + p_4 z^{m-3} + \dots + p_m z\} \\ &= (p_1^2 - p_2) z^{m-1} + (p_1 p_2 - p_3) z^{m-2} + \dots \\ &\qquad + (p_1 p_{m-1} - p_m) z + p_1 p_m. \end{aligned}$$

Proceeding in this manner we can express all powers of z higher than the $m-1$th in terms of the p's and powers of z lower than the mth, thus reducing $\chi(z)$ to a function of the $m-1$th degree in z.

As an example, let $m = 2$, and write

$$\phi(z) \equiv z^2 + p_1 z + p_2,$$

and let us suppose that

$$\chi(z) \equiv D_0 z^3 + D_1 z^2 + D_2 z + D_3$$

vanishes for both roots of $\phi(z) = 0$.

We have $\qquad z^2 = -(p_1 z + p_2),$

consequently $\qquad z^3 = (p_1^2 - p_2) z + p_1 p_2;$

introducing these values of z^3 and z^2 into $\chi(z)$, we obtain

$$\chi(z) = \{D_0(p_1^2 - p_2) - D_1 p_1 + D_2\} z + \{D_0 p_1 p_2 - D_1 p_2 + D_3\},$$

and since $\chi(z)$ by hypothesis vanishes for both roots of $\phi(z) = 0$, we must have

$$\left.\begin{aligned} D_0(p_1^2 - p_2) - D_1 p_1 + D_2 = 0 \\ D_0 p_1 p_2 - D_1 p_2 + D_3 = 0 \end{aligned}\right\};$$

equations which express the conditions that $\phi(z)$ may be a factor of $\chi(z)$.

3. We now turn to the consideration of two important operators of which we shall make large use in what follows.

If we write

$$\Sigma \frac{d}{dz_1} = \frac{d}{dz_1} + \frac{d}{dz_2} + \frac{d}{dz_3} + \ldots + \frac{d}{dz_m},$$

and denote this operation by $-\delta$, so that

$$\Sigma \frac{d}{dz_1} = -\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

it is clear that if F denote any function of the p's, we have

$$\frac{dF}{dz_1} = \frac{dF}{dp_1}\frac{dp_1}{dz_1} + \frac{dF}{dp_2}\frac{dp_2}{dz_1} + \frac{dF}{dp_3}\frac{dp_3}{dz_1} + \&c.;$$

hence $$\Sigma \frac{d}{dz_1} F = \frac{dF}{dp_1} \Sigma \frac{d}{dz_1} p_1 + \frac{dF}{dp_2} \Sigma \frac{d}{dz_1} p_2 + \frac{dF}{dp_3} \Sigma \frac{d}{dz_1} p_3 + \ldots,$$

or $$\delta F = \frac{dF}{dp_1}\delta p_1 + \frac{dF}{dp_2}\delta p_2 + \frac{dF}{dp_3}\delta p_3 + \&c. \ldots\ldots\ldots\ldots\ldots (2).$$

If now z stand for any one of the m z's we have $\phi(z) \equiv 0$, and it is obvious that if we operate on $\phi(z)$ with δ the result will be zero. Performing then this operation on

$$z^m + p_1 z^{m-1} + p_2 z^{m-2} + \ldots + p_m = 0,$$

we obtain

$$\begin{aligned} -mz^{m-1} - (m-1)p_1 z^{m-2} - (m-2)p_2 z^{m-3} - \ldots - p_{m-1} \\ + z^{m-1}\delta p_1 + z^{m-2}\delta p_2 + \ldots + \delta p_m = 0, \end{aligned}$$

or $$\begin{aligned} z^{m-1}(\delta p_1 - m) + z^{m-2}(\delta p_2 - (m-1)p_1) + \\ z^{m-3}(\delta p_3 - (m-2)p_2) + \ldots + \delta p_m - p_{m-1} = 0 \ldots (3). \end{aligned}$$

Now since this expression is of the $(m-1)$th degree in z and vanishes for each root of $\phi(z) = 0$, the coefficient of each power of z as well as the last term must vanish by Art. 2. Hence we find

$$\left.\begin{aligned} \delta p_1 = m;\ \delta p_2 = (m-1)p_1;\ \delta p_3 = (m-2)p_2; \\ \delta p_r = (m+1-r)p_{r-1};\ \&c.;\ \delta p_m = p_{m-1}; \end{aligned}\right\} \ldots\ldots (4),$$

r being any integer from 1 to m.

If we now let

$$z = x/y\,;\quad z_1 = x_1/y_1\,;\quad z_2 = x_2/y_2\,;\quad \&c.\,;\quad z_r = x_r/y_r,$$

and denote the product $y_1 y_2 y_3 \ldots y_m$ by p_0, we find, on introducing these values into the identity

$$\phi(z) \equiv (z - z_1)(z - z_2)(z - z_3)\ldots(z - z_m),$$

and multiplying by $p_0 y^m$,

$$\left.\begin{aligned} p_0 y^m \phi(z) &\equiv p_0 x^m + p_0 p_1 x^{m-1} y + p_0 p_2 x^{m-2} y^2 \\ &\qquad + p_0 p_3 x^{m-3} y^3 + \ldots + p_0 p_m \\ &\equiv (xy_1 - yx_1)(xy_2 - yx_2)(xy_3 - yx_3)\ldots \\ &\qquad\qquad (xy_m - yx_m) \\ &\equiv u(x, y), \text{ say,} \end{aligned}\right\} \ldots\ldots(5).$$

The operator δ now becomes

$$-\Sigma y_1 \frac{d}{dx_1},$$

and it follows, by a similar process of reasoning to that already employed in the determination of δp_1, δp_2, &c., that if we write $u(x, y)$ with binomial coefficients, so that

$$u(x, y) \equiv a_0 x^m + m a_1 x^{m-1} y + \frac{m(m-1)}{1.2} a_2 x^{m-2} y^2 + \ldots + a_m y^m \ldots(6),$$

we must have

$$\delta a_0 = 0\,;\quad \delta a_1 = a_0\,;\quad \delta a_r = r a_{r-1} \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

r being any integer from 1 to m.

We shall now investigate the properties of another operator which we shall denote by Δ, and which we define as follows:

$$\Delta = -\Sigma x_1 \frac{d}{dy_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Now if z or x/y stand for any one of the roots of $\phi(z) = 0$, we have $u(x, y) \equiv 0$, or

$$p_0 x^m + (p_0 p_1) x^{m-1} y + (p_0 p_2) x^{m-2} y^2 + \ldots + (p_0 p_m) y^m = 0\,;$$

operating with Δ on the above equation and bearing in mind that $(p_0 p_m) = (-1)^m x_1 x_2 x_3 \ldots x_m$, and that consequently $\Delta(p_0 p_m) = 0$, we have

$$\begin{aligned} x^m \Delta p_0 + x^{m-1} y \Delta(p_0 p_1) + x^{m-2} y^2 \Delta(p_0 p_2) + \ldots + x y^{m-1} \Delta(p_0 p_{m-1}) \\ - (p_0 p_1) x^m - 2x^{m-1} y (p_0 p_2) - 3x^{m-2} y^2 (p_0 p_3) - \ldots \\ - m x y^{m-1} (p_0 p_m) \equiv 0\,; \end{aligned}$$

hence, collecting the coefficients of the various powers of x and dividing by xy^{m-1}, we obtain on replacing x/y by z

$$z^{m-1}\{\Delta p_0 - (p_0p_1)\} + z^{m-2}\{\Delta(p_0p_1) - 2(p_0p_2)\}$$
$$+ z^{m-3}\{\Delta(p_0p_2) - 3(p_0p_3)\} + \ldots + \Delta(p_0p_{m-1}) - (p_0p_m) \equiv 0 \ldots (9);$$

we are therefore led, by the rule laid down in Art. 2, to the equations

$$\left.\begin{aligned} \Delta p_0 &= p_0p_1, \\ \Delta(p_0p_1) &= 2p_0p_2, \\ \Delta(p_0p_2) &= 3p_0p_3, \\ \Delta(p_0p_r) &= (r+1)\,p_0p_{r+1}, \\ \Delta(p_0p_{m-1}) &= p_0p_m, \\ \Delta(p_0p_m) &= 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (10),$$

r having any integer value from 0 to $m-1$.

Again, since

$$\Delta = -\Sigma x_1 \frac{d}{dy_1},$$

it is obvious that

$$\Delta(p_0p_r) = p_0\Delta p_r + p_r\Delta p_0;$$

and we hence arrive readily at the following equations:

$$\left.\begin{aligned} &\Delta p_0 = p_0p_1;\ \ \Delta p_1 = 2p_2 - p_1^2;\ \ \Delta p_2 = 3p_3 - p_1p_2; \\ &\Delta p_3 = 4p_4 - p_1p_3;\ \ \Delta p_r = (r+1)\,p_{r+1} - p_1p_r;\ \ \Delta p_m = -p_1p_m; \end{aligned}\right\} \ldots (11)$$

r having any integer value from 0 to $m-1$.

If $u(x, y)$ were written with binomial coefficients we should find

$$\Delta a_r = (m-r)\,a_{r+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

We have also, as may be readily proved,

$$\Delta F = \frac{dF}{dp_1}\Delta p_1 + \frac{dF}{dp_2}\Delta p_2 + \frac{dF}{dp_3}\Delta p_3 + \&c. \ldots\ldots\ldots (13),$$

F being any function of the p's.

4. Adopting the usual notation for the sum of similar powers of the roots of $\phi(z) = 0$, we write

$$s_r \equiv \Sigma z_1^r = \Sigma\left(\frac{x}{y}\right)^r \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and we proceed to show how all such sums may be obtained by an operative process alone.

Since $\Delta \equiv -\Sigma x_1 \dfrac{d}{dy_1}$, it is clear we have in all cases

$$\Delta s_r \equiv r s_{r+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Operating then on the identical equation

$$s_1 = -p_1,$$

we obtain $$\Delta s_1 = -\Delta p_1,$$

or $$s_2 = -\Delta p_1 = p_1^2 - 2p_2, \text{ by Art. 3} \quad \ldots\ldots\ldots\ldots(3).$$

Operating again on this equation with Δ, we obtain

$$\Delta s_2 = 2p_1 \Delta p_1 - 2\Delta p_2,$$

or $$2s_3 = 2p_1 \Delta p_1 - 2\Delta p_2,$$

or $$s_3 = -p_1^3 + 3p_1 p_2 - 3p_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4);$$

operating again on the above with Δ, we find

$$s_4 = p_1^4 - 4p_1^2 p_2 + 2p_2^2 - 4p_4 \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

By operating on this equation we obtain the value of s_5, and thus by successive applications of the operator Δ we obtain s_r in terms of $p_1, p_2, p_3, \ldots p_r$, r being any positive integer equal to or less than m.

In this manner s_m is found, and by operating on the value thus obtained and bearing in mind that $\Delta p_m = -p_1 p_m$ we can arrive at the value of s_{m+1}, and thus, by successive applications of Δ, at the value of s_{m+r}, r being any positive integer whatever.

The value of s_{-r} can at once be obtained from that of s_r by means of the transformation

$$\left.\begin{array}{lcl} z & \text{into} & z^{-1} \\ p_r & \text{,,} & \dfrac{p_{m-r}}{p_m} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6);$$

or it can be derived by successive applications of the operator δ on the equation

$$s_{-1} = -\frac{p_{m-1}}{p_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

for, since

$$\delta s_{-1} = -\Sigma \frac{d}{dz_1} \Sigma \frac{1}{z_1} = \Sigma \frac{1}{z_1^2} = s_{-2}, \quad \delta s_{-2} = 2s_{-3}, \quad \&c.,$$

it follows that s_{-r} can be obtained by an operative process on the above equation.

It will be seen at once that if we wished to make use of the operator δ in obtaining the value of s_r, we could first find, by means of that operator, the value of s_{-r}, and then by the aid of the above transformation deduce the value of s_r.

Newton's formulæ for the sums of the powers of the z's in terms of the p's can be immediately obtained by an operative process in the following manner.

We have $$p_0 s_1 + p_0 p_1 = 0 \quad \text{.....(8)},$$
and if we operate on this equation with Δ, bearing in mind that $\Delta p_0 = p_0 p_1$, $\Delta p_0 p_r = (r+1)\, p_0 p_{r+1}$, we find
$$p_0 s_2 + s_1 p_0 p_1 + 2 p_0 p_2 = 0\,;$$
operating again on this equation, we obtain
$$2 p_0 s_3 + s_2 p_0 p_1 + s_1 \,.\, 2 p_0 p_2 + s_2 p_0 p_1 + 6 p_0 p_3 = 0,$$
or $$p_0 s_3 + s_2 p_0 p_1 + s_1 p_0 p_2 + 3 p_0 p_3 = 0 \quad \text{.....(9)};$$
a second application of Δ gives us, on dividing by 3,
$$p_0 s_4 + s_3 p_0 p_1 + s_2 p_0 p_2 + s_1 p_0 p_3 + 4 p_0 p_4,$$
and so it follows that by successive applications of Δ on the above equation all the Newtonian equations can be derived.

From these equations the following determinant expressions for $s_2, s_3, \ldots s_r$ readily follow.

$$\left.\begin{aligned} s_2 &= \begin{vmatrix} -p_1 & 1 \\ 2p_2 & -p_1 \end{vmatrix}; \quad s_3 = \begin{vmatrix} -p_1 & 1 & 0 \\ 2p_2 & -p_1 & 1 \\ -3p_3 & p_2 & -p_1 \end{vmatrix}; \\ s_4 &= \begin{vmatrix} -p_1 & 1 & 0 & 0 \\ 2p_2 & -p_1 & 1 & 0 \\ -3p_3 & p_2 & -p_1 & 1 \\ 4p_4 & -p_3 & p_2 & -p_1 \end{vmatrix}; \quad \&c. \end{aligned}\right\} \ldots(10).$$

5. The values of $p_1, p_2, p_3, \ldots p_r$ may all be obtained by an operative process in terms of $s_1, s_2, s_3, \ldots s_r$.

We have $p_0 p_1 = -p_0 s_1$; operating on this equation with Δ we obtain, writing $\Delta p_0 = -p_0 s_1$,
$$2 p_0 p_2 = p_0 (s_1^2 - s_2) \quad \text{.....(1)};$$
operating again on this equation we find
$$2 \,.\, 3 \,.\, p_0 p_3 = p_0 \{-s_1^3 + 3 s_1 s_2 - 2 s_3\} \quad \text{.....(2)};$$
another application of the operator gives us
$$2 \,.\, 3 \,.\, 4 \,.\, p_0 p_4 = p_0 \{s_1^4 - 6 s_1^2 s_2 + 8 s_1 s_3 + 3 s_2^2 - 6 s_4\} \quad \text{....(3)}.$$

Thus the value of p_r in terms of $s_1, s_2, \ldots s_r$ is obtained by successive applications of Δ.

The following determinant expressions are easily obtained, viz.

$$2p_2 = \begin{vmatrix} s_1 & 1 \\ s_2 & s_1 \end{vmatrix}; \quad 2.3.p_3 = -\begin{vmatrix} s_1 & 1 & 0 \\ s_2 & s_1 & 2 \\ s_3 & s_2 & s_1 \end{vmatrix};$$

$$2.3.4.p_4 = \begin{vmatrix} s_1 & 1 & 0 & 0 \\ s_2 & s_1 & 2 & 0 \\ s_3 & s_2 & s_1 & 3 \\ s_4 & s_3 & s_2 & s_1 \end{vmatrix}; \quad \&c.$$

6. *Every rational and integral symmetric function of the roots of the equation* $\phi(z)=0$ *can be expressed as a rational and integral function of the coefficients of this equation by means of the operative process* Δ.

If, then, Σ denote any such symmetric function of the z's, we shall show how Σ can be expressed by means of an equation

$$\Sigma = F,$$

F denoting a rational and integral function of the p's. The sum of the exponents of all the z's which enter into any term of Σ is called the *degree* of that function. Thus, for example, if $\Sigma z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_r^{\lambda_r}$ be the function considered, $\lambda_1, \lambda_2, \ldots \lambda_r$ being integers and written in descending order of magnitude, the *degree* of Σ will be $\lambda_1 + \lambda_2 + \ldots + \lambda_r$. We shall use the term *class* to denote the highest degree in which a root enters into a term of Σ.

Thus, considering the above form of Σ, its class would be λ_1. It is only stating the same thing in another way to say that if we write $z = x/y$, the class of Σ is equal to the number of times we must multiply Σ by p_0 in order to clear it of fractions. Thus if we had to multiply Σ by p_0^3 in order to clear it of fractions its *class* would be three. Symmetric functions may be arranged according to their degree, but symmetric functions of a given degree may be of different classes. Thus $\Sigma z_1 z_2 z_3 z_4$, $\Sigma z_1^2 z_2 z_3$, $\Sigma z_1^2 z_2^2$, $\Sigma z_1^3 z_2$, Σz_1^4 are all of the fourth degree, but the class of the first is one, that of the second two, of the third two, of the fourth three, and of the fifth four. If we now consider any function of the p's, the term *order* of that function is used to denote the number of factors in its highest term.

Thus, if any term were $p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} \ldots p_r^{\mu_r}$, the order of the function, supposing this to be its highest term, would be

$$\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_r.$$

The term *weight* is employed to denote the sum of the suffixes attached to each factor of a term occurring in a function of the p's. Thus the weight of the term already considered is

$$\mu_1 + 2\mu_2 + 3\mu_3 + \ldots + r\mu_r.$$

In all the functions which we shall consider the weight of each term will be the same and will be called the weight of the function. We shall employ the term *rank* to denote the number of roots which enter into each term of a symmetric function. Returning now to the consideration of symmetric functions we observe that every symmetric function of the z's may be regarded as a primary form from which a secondary form may be derived by substituting for z its value x/y and then clearing of fractions by multiplying by $p_0^{\lambda_1}$, where λ_1 is the class of the function, so that the secondary form σ is defined by the equation

$$\sigma = p_0^{\lambda_1}\Sigma.$$

These two forms, Σ and σ, will give us different results when operated upon by Δ and will conduct us to many important conclusions. We shall call the class of Σ, λ_1, its degree n and its rank r. The class of σ will be λ_1, its degree n and the number of roots entering into each term r.

Any primary symmetric function when operated upon by Δ will give rise to a sum of symmetric functions each term of which will involve r roots of $\phi(z) = 0$. These symmetric functions will all be of the $(n+1)$th degree and the class of *one* will be $\lambda_1 + 1$ and that of the remaining functions λ_1. Any secondary symmetric function when operated upon by Δ will give rise to a sum of secondary symmetric functions all of the degree $(n+1)$ and all of the class λ_1. Each term of some of these symmetric functions will, however, involve $(r+1)$ roots. These theorems can easily be verified by assigning a certain definite form to Σ. Thus, let us give to the Σ the form $\Sigma z_1^4 z_2^2 z_3$ and operate with Δ; we have then $\Delta\Sigma = 4\Sigma z_1^5 z_2^2 z_3 + 2\Sigma z_1^4 z_2^3 z_3 + 2\Sigma z_1^4 z_2^2 z_3^2$.

We see then that $\Delta\Sigma$ consists of three symmetric functions of the eighth degree, each term of these functions involves the same number of roots as that involved in a term of Σ, the class of the first symmetric function being 5 and that of the remaining two being 4. The class of Σ being 4 its secondary form σ will be given by the equation $\sigma = p_0^4\Sigma$, or

$$\sigma = \Sigma x_1^4 x_2^2 y_2^2 x_3 y_3^3 y_4^4 \ldots y_m^4;$$