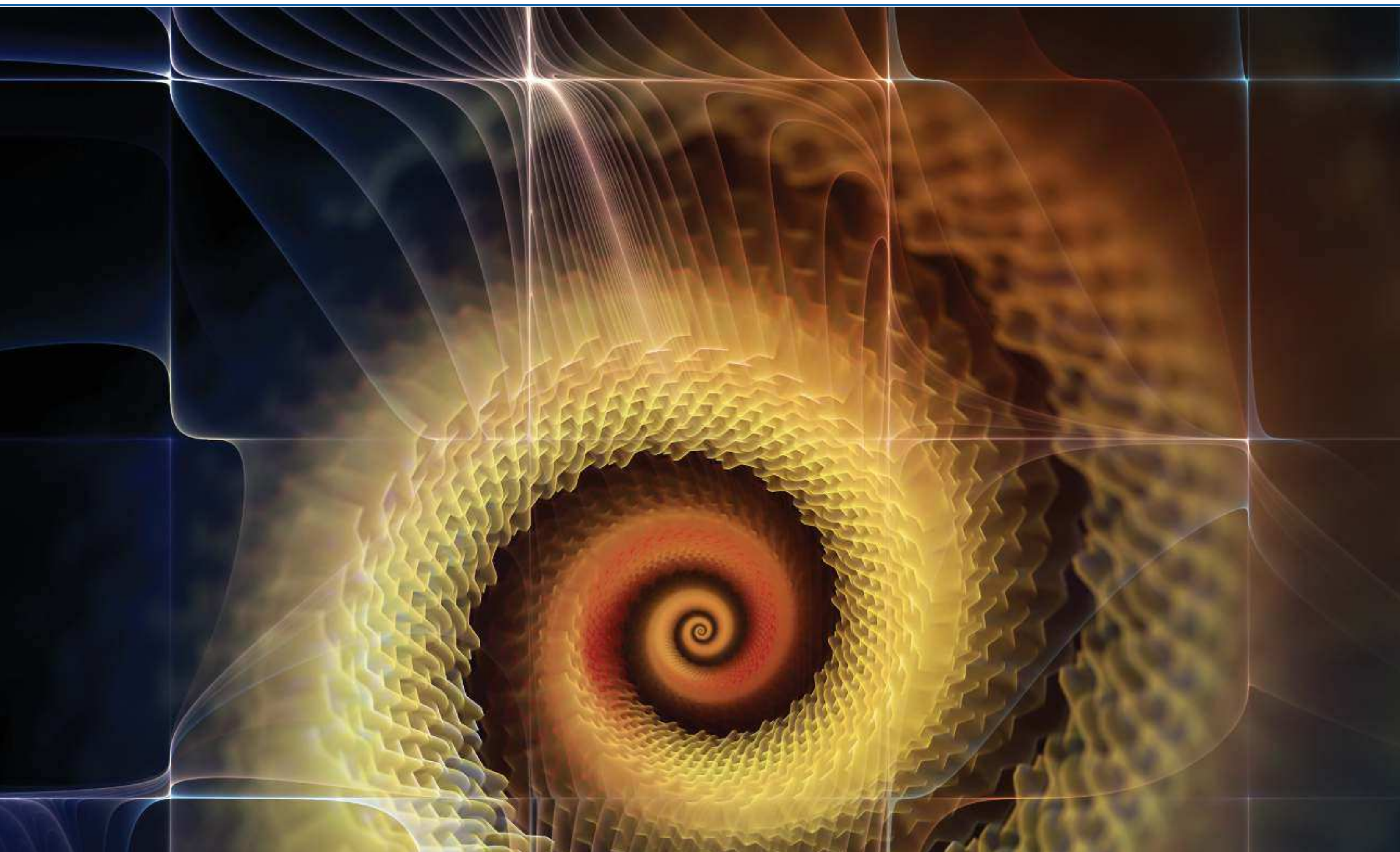
Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information



Chapter 1
Polynomials

This chapter is about polynomials, which include linear and quadratic expressions. When you have completed it, you should

- be able to add, subtract, multiply and divide polynomials
- understand the words 'quotient' and 'remainder' used in dividing polynomials
- be able to use the method of equating coefficients
- be able to use the remainder theorem and the factor theorem.

Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information

1.1 Polynomials

You already know a good deal about polynomials from your work on quadratics in Chapter 4 of *Pure Mathematics 1* (unit P1), because a quadratic is a special case of a polynomial. Here are some examples of polynomials.

$$3x^2 - 2x^2 + 1 \qquad 3 \qquad 4 - 2x \qquad x^2 \qquad 1$$
$$2x^4 \qquad 1 - 2x + 3x^5 \qquad \sqrt{2}x^2 \qquad \frac{1}{2}x^{17} \qquad x$$

A (non-zero) polynomial**, p(x) is an expression in x of the form**

$$ax^n + bx^{n-1} + \ldots + jx + k$$

where a, b, c, ..., k are real numbers, $a \neq 0$, and n is a non-negative integer.

The number n is called the **degree** of the polynomial. The expressions ax^n, bx^{n-1}, ..., jx and k which make up the polynomial are called **terms**. The numbers a, b, c, ..., j and k are called **coefficients**; a is the **leading coefficient**. The coefficient k is the **constant term**.

Thus, in the quadratic polynomial $4x^2 - 3x + 1$, the degree is 2; the coefficients of x^2 and x, and the constant term, are 4, -3 and 1 respectively.

Polynomials with low degree have special names: if the polynomial has

- degree 0 it is called a **constant polynomial**, or a **constant**
- degree 1 it is called a **linear polynomial**
- degree 2 it is called a **quadratic polynomial**, or a **quadratic**
- degree 3 it is called a **cubic polynomial**, or a **cubic**
- degree 4 it is called a **quartic polynomial**, or a **quartic**.

When a polynomial is written as $ax^n + bx^{n-1} + \ldots + jx + k$, with the term of highest degree first and the other terms in descending degree order finishing with the constant term, the terms are said to be in **descending order**. If the terms are written in the reverse order, they are said to be in **ascending order** (or ascending powers of x). For example, $3x^4 + x^2 - 7x + 5$ is in descending order; in ascending order it is $5 - 7x + x^2 + 3x^4$. It is the same polynomial whatever order the terms are written in.

The functions $\frac{1}{x} = x^{-1}$ and $\sqrt{x} = x^{\frac{1}{2}}$ are not polynomials, because the powers of x are not positive integers or zero.

Polynomials have much in common with integers. You can add them, subtract them and multiply them together and the result is another polynomial. You can even divide a polynomial by another polynomial, as you will see in Section 1.4.

1.2 Addition, subtraction and multiplication of polynomials

To add or subtract two polynomials, you simply add or subtract the coefficients of corresponding powers; in other words, you collect like terms. Suppose that you want to add $2x^3 + 3x^2 - 4$ to $x^2 - x - 2$, then you can set out the working like this:

$$\begin{array}{rcrcrcr}
2x^2 & + & 3x^2 & & & - & 4 \\
 & & x^2 & - & x & - & 2 \\
\hline
2x^2 & + & 4x^2 & - & x & - & 6
\end{array}$$

Notice that you must leave gaps in places where the coefficient is zero. You need to do addition so often that it is worth getting used to setting out the work in a line, thus:

$$\begin{aligned}
(2x^3+3x^2-4)+(x^2-x-2) &= (2+0)x^3+(3+1)x^2+(0+(-1))x+((-4)+(-2)) \\
&= 2x^3+4x^2-x-6.
\end{aligned}$$

You will soon find that you can miss out the middle step and go straight to the answer.

The result of the polynomial calculation $(2x^3+3x^2-4)-(2x^3+3x^2-4)$ is 0. This is a special case, and it is called the **zero polynomial**. It has no degree.

Look back at the definition of a polynomial, and see why the zero polynomial was not included there.

Multiplying polynomials is harder. It relies on the rules for multiplying out brackets:

$a(b+c+\ldots+k) = ab+ac+\ldots+ak$ and $(b+c+\ldots+k)a = ba+ca+\ldots+ka$.

To apply these rules to multiplying the two polynomials $5x+3$ and $2x^2-5x+1$, replace $2x^2-5x+1$ for the time being by z. Then

$$\begin{aligned}
(5x+3)(2x^2-5x+1) &= (5x+3)z \\
&= 5xz+3z \\
&= 5x(2x^2-5x+1)+3(2x^2-5x+1) \\
&= (10x^3-25x^2+5x)+(6x^2-15x+3) \\
&= 10x^3-19x^2-10x+3.
\end{aligned}$$

In practice, it is easier to note that every term in the left bracket multiplies every term in the right bracket. You can show this by setting out the steps in the following way.

$$\begin{array}{rcrcrcr|r}
 & & 2x^2 & - & 5x & + & 1 & \times \\
\hline
10x^3 & - & 25x^2 & + & 5x & & & 5x \\
 & + & 6x^2 & - & 15x & + & 3 & +3 \\
\hline
10x^3 & + & (-25+6)x^2 & + & (5-15)x & + & 3 &
\end{array}$$

giving the result $10x^3-19x^2-10x+3$.

It is worth learning to work horizontally. The arrows below show the term $5x$ from the first bracket multiplied by $-5x$ from the second bracket to get $-25x^2$.

$$\begin{aligned}
(5x+3)(2x^2-5x+1) &= 5x(2x^2-5x+1)+3(2x^2-5x+1) \\
&= (10x^3-25x^2+5x)+(6x^2-15x+3) \\
&= 10x^3-19x^2-10x+3.
\end{aligned}$$

You could shorten the process and write

$$\begin{aligned}
(5x+3)(2x^2-5x+1) &= 10x^3-25x^2+5x+6x^2-15x+3 \\
&= 10x^3-19x^2-10x+3.
\end{aligned}$$

Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information

If you multiply a polynomial of degree m by a polynomial of degree n, you have a calculation of the type

$$\left(ax^m + bx^{m-1} + \ldots\right)\left(Ax^m + Bx^{m-1} + \ldots\right) = aAx^{m+n} + \ldots$$

in which the largest power of the product is $m + n$. Also, the coefficient aA is not zero because neither of a and A is zero. This shows that:

When you multiply two polynomials, the degree of the product polynomial is the sum of the degrees of the two polynomials.

Exercise 1A

1 State the degree of each of the following polynomials.

a $x^3 - 3x^2 + 2x - 7$ **b** $5x + 1$ **c** $8 + 5x - 3x^2 + 7x + 6x^4$

d 3 **e** $3 - 5x$ **f** x^0

2 In each part find $p(x) + q(x)$, and give your answer in descending order.

a $p(x) = 3x^2 + 4x - 1$, $q(x) = x^2 + 3x + 7$

b $p(x) = 4x^3 + 5x^2 - 7x + 3$, $q(x) = x^3 - 2x^2 + x - 6$

c $p(x) = 3x^4 - 2x^3 + 7x^2 - 1$, $q(x) = -3x - x^3 + 5x^4 + 2$

d $p(x) = 2 - 3x^3 + 2x^5$, $q(x) = 2x^4 + 3x^3 - 5x^2 + 1$

e $p(x) = 3 + 2x - 4x^2 - x^3$, $q(x) = 1 - 7x + 2x^2$

3 For each of the pairs of polynomials given in Question 2 find $p(x) - q(x)$.

4 Note that $p(x) + p(x)$ may be shortened to $2p(x)$. Let $p(x) = x^3 - 2x^2 + 5x - 3$ and $q(x) = x^2 - x + 4$. Express each of the following as a single polynomial.

a $2p(x) + q(x)$ **b** $3p(x) - q(x)$

c $p(x) - 2q(x)$ **d** $3p(x) - 2q(x)$

5 Find the following polynomial products.

a $(2x - 3)(3x + 1)$ **b** $(x^2 + 3x - 1)(x - 2)$

c $(x^2 + x - 3)(2x + 3)$ **d** $(3x - 1)(4x^2 - 3x + 2)$

e $(x^2 + 2x - 3)(x^2 + 1)$ **f** $(2x^2 - 3x + 1)(4x^2 + 3x - 5)$

g $(x^3 + 2x^2 - x + 6)(x + 3)$ **h** $(x^3 - 3x^2 + 2x - 1)(x^2 - 2x - 5)$

i $(1 + 3x - x^2 + 2x^3)(3 - x + 2x^2)$ **j** $(2 - 3x + x^2)(4 - 5x + x^3)$

k $(2x + 1)(3x - 2)(x + 5)$ **l** $(x^2 + 1)(x - 3)(2x^2 - x + 1)$

6 In each of the following products find the coefficient of x and the coefficient of x^2.

a $(x + 2)(x^2 - 3x + 6)$ **b** $(x - 3)(x^2 + 2x - 5)$

c $(2x + 1)(x^2 - 5x + 1)$ **d** $(3x - 2)(x^2 - 2x + 7)$

e $(2x - 3)(3x^2 - 6x + 1)$ **f** $(2x - 5)(3x^3 - x^2 + 4x + 2)$

g $(x^2 + 2x - 3)(x^2 + 3x - 4)$ **h** $(3x^2 + 1)(2x^2 - 5x + 3)$

i $(x^2 + 3x - 1)(x^3 + x^2 - 2x + 1)$ **j** $(3x^2 - x + 2)(4x^3 - 5x + 1)$

Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information

7 In each of the following the product of $Ax + B$ with another polynomial is given. Using the fact that A and B are constants, find A and B.

a $(Ax+B)(x-3) = 4x^2 - 11x - 3$

b $(Ax+B)(x+5) = 2x^2 + 7x - 15$

c $(Ax+B)(3x-2) = 6x^2 - x - 2$

d $(Ax+B)(2x+5) = 6x^2 + 11x - 10$

e $(Ax+B)(x^2-1) = x^3 + 2x^2 - x - 2$

f $(Ax+B)(x^2+4) = 2x^3 - 3x^2 + 8x - 12$

g $(Ax+B)(2x^2-3x+4) = 4x^3 - x + 12$

h $(Ax+B)(3x^2-2x-1) = 6x^3 - 7x^2 + 1$

1.3 Equations and identities

In this chapter so far you have learned how to add, subtract and multiply polynomials, and you can now carry out calculations such as

$$(2x+3)+(x-2)=3x+1$$
$$(x^2-3x-4)-(2x+1)=x^2-5x-5 \quad \text{and}$$
$$(1-x)(1+x+x^2)=1-x^3$$

fairly automatically.

However, you should realise that these are not equations in the normal sense, because they are true for all values of x.

In P1 Section 10.6, you saw that when two expressions take the same values for every value of the variable, they are said to be **identically equal**, and a statement such as

$$(1-x)(1+x+x^2)=1-x^3$$

is called an **identity**.

To emphasise that an equation is an identity, the symbol $\equiv$ is used. The statement $(1-x)(1+x+x^2) \equiv 1-x^3$ means that $(1-x)(1+x+x^2)$ and $1-x^3$ are equal for all values of x.

But now suppose that $Ax + B \equiv 2x + 3$. What can you say about A and B?
As $Ax + B \equiv 2x + 3$ is an identity, it is true for all values of x. In particular, it is true for $x = 0$. Therefore $A \times 0 + B = 2 \times 0 + 3$, giving $B = 3$. But the identity is also true when $x = 1$, so $A \times 1 + 3 = 2 \times 1 + 3$, giving $A = 2$. Therefore:

If $Ax + B \equiv 2x + 3$, then $A = 2$ and $B = 3$.

This is an example of the process called **equating coefficients**. The full result is:

> **If $ax^n + bx^{n-1} + \ldots + k \equiv Ax^n + Bx^{n-1} + \ldots + K$,**
>
> **then $a = A$, $b = B$, …, $k = K$.**

The statement in the box says that, if two polynomials are equal for all values of x, then all the coefficients of corresponding powers of x are equal.

This result may not surprise you, but you should be aware that you are using it. Indeed, it is very likely that you have used it before now without being aware of it.

CAMBRIDGE
Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information

EXAMPLE 1.3.1

One factor of $3x^2 - 5x - 2$ is $x - 2$. Find the other factor.

There is nothing wrong in writing down the answer by inspection as $3x + 1$. But the process behind this quick solution is as follows.

Suppose that the other factor is $Ax + B$. Then $(Ax + B)(x - 2) \equiv 3x^2 - 5x - 2$, and, multiplying out, you get

$$Ax^2 + (-2A + B)x - 2B \equiv 3x^2 - 5x - 2.$$

By equating coefficients of x^2, you get $A = 3$. Equating coefficients of x^0, the constant term, you get $-2B = -2$, giving $B = 1$. Therefore the other factor is $3x + 1$.

You can also check that the middle term, $-2A + B = -6 + 1 = -5$, is correct.

You should continue to write down the other factor by inspection if you can. However, in some cases, it is not easy to see what the answer will be without intermediate working.

EXAMPLE 1.3.2

If $4x^3 + 2x^2 + 3 \equiv (x - 2)(Ax^2 + Bx + C) + R$, find A, B, C and R.

Multiplying out the right side gives

$$4x^3 + 2x^2 + 3 \equiv Ax^3 + (-2A + B)x^2 + (-2B + C)x + (-2C + R).$$

Equating coefficients of x^3: $4 = A$.

Equating coefficients of x^2: $2 = -2A + B = -2 \times 4 + B = -8 + B$, so $B = 10$.

Equating coefficients of x: $0 = -2B + C = -20 + C$, so $C = 20$.

Equating coefficients of x^0: $3 = -2C + R = -40 + R$, giving $R = 43$.

Therefore $A = 4$, $B = 10$, $C = 20$ and $R = 43$, so

$$4x^3 + 2x^2 + 3 \equiv (x - 2)(4x^2 + 10x + 20) + 43.$$

In practice, people often use the symbol for equality, =, when they really mean the symbol for identity, ≡. The context usually suggests which meaning is intended.

Exercise 1B

1 In each of the following quadratic polynomials one factor is given. Find the other factor.

a $x^2 + x - 12 \equiv (x + 4)(\quad)$ **b** $x^2 + 14x - 51 \equiv (x - 3)(\quad)$

c $3x^2 + 5x - 22 \equiv (x - 2)(\quad)$ **d** $35x^2 + 48x - 27 \equiv (5x + 9)(\quad)$

e $2x^2 - x - 15 \equiv (2x + 5)(\quad)$ **f** $14x^2 + 31x - 10 \equiv (2x + 5)(\quad)$

2 In each of the following identities find the values of A, B and R.

a $x^2 - 2x + 7 \equiv (x + 3)(Ax + B) + R$

b $x^2 + 9x - 3 \equiv (x + 1)(Ax + B) + R$

Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information

c $15x^2 - 14x - 8 \equiv (5x + 2)(Ax + B) + R$

d $6x^2 + x - 5 \equiv (2x + 1)(Ax + B) + R$

e $12x^2 - 5x + 2 \equiv (3x - 2)(Ax + B) + R$

f $21x^2 - 11x + 6 \equiv (3x - 2)(Ax + B) + R$

3 In each of the following identities find the values of A, B, C and R.

a $x^3 - x^2 - x + 12 \equiv (x + 2)(Ax^2 + Bx + \mathrm{C}) + R$

b $x^3 - 5x^2 + 10x + 10 \equiv (x - 3)(Ax^2 + Bx + C) + R$

c $2x^3 + x^2 - 3x + 4 \equiv (2x - 1)(Ax^2 + Bx + C) + R$

d $12x^3 + 11x^2 - 7x + 5 \equiv (3x + 2)(Ax^2 + Bx + C) + R$

e $4x^3 + 4x^2 - 37x + 5 \equiv (2x - 5)(Ax^2 + Bx + C) + R$

f $9x^3 + 12x^2 - 15x - 10 \equiv (3x + 4)(Ax^2 + Bx + C) + R$

4 In each of the following identities find the values of A, B, C, D and R.

a $2x^4 + 3x^3 - 5x^2 + 11x - 5 \equiv (x + 3)(Ax^3 + Bx^2 + Cx + D) + R$

b $4x^4 - 7x^3 - 2x^2 - 2x + 7 \equiv (x - 2)(Ax^3 + Bx^2 + Cx + D) + R$

c $6x^4 + 5x^3 - x^2 + 3x + 2 \equiv (2x + 1)(Ax^3 + Bx^2 + Cx + D) + R$

d $3x^4 - 7x^3 + 17x^2 - 14x + 5 \equiv (3x - 1)(Ax^3 + Bx^2 + Cx + D) + R$

1.4 Division of polynomials

You can, if you wish, carry out division of polynomials using a layout like the one for long division of integers. You might already have seen and used such a process, and examples are given below. However, you can also use the method of equating coefficients for division, as will now be explained.

When you divide 112 by 9, you get an answer of 12 with 4 over. The number 9 is called the divisor, 12 is the quotient and 4 the remainder. You can express this as an equation in integers, $112 = 9 \times 12 + 4$. The remainder r has to satisfy the inequality $0 \leq r < 9$.

Now look back at Example 1.3.2. You will see that it is an identity of just the same shape, but with polynomials instead of integers. So you can say that, when $4x^3 + 2x^2 + 3$ is divided by the divisor $x - 2$, the quotient is $4x^2 + 10x + 20$ and the remainder is 43. The degree of the remainder (in this case 0) has to be less than the degree of the divisor. The degree of the quotient $4x^2 + 10x + 20$, which is 2, is equal to the difference between the degree of the polynomial $4x^3 + 2x^2 + 3$, which is 3, and the degree of the divisor $x - 2$, which is 1.

When a polynomial, a(x), is divided by a non-constant divisor, **b(x), the** quotient **q(x) and the** remainder **r(x) are defined by the identity**

$$\mathbf{a}(x) \equiv \mathbf{b}(x)\mathbf{q}(x) + \mathbf{r}(x),$$

where the degree of the remainder is less than the degree of the divisor.

The degree of the quotient is equal to the degree of a(x) – the degree of b(x).

CAMBRIDGE

Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information

EXAMPLE 1.4.1

Find the quotient and remainder when $x^4 + x + 2$ is divided by $x + 1$.

Equating coefficients approach

Using the result in the box, as the degree of $x^4 + x + 2$ is 4 and the degree of $x + 1$ is 1, the degree of the quotient is $4 - 1 = 3$. And as the degree of the remainder is less than 1, the remainder is a constant.

Let the quotient be $Ax^3 + Bx^2 + Cx + D$, and let the remainder be R. Then

$$x^4 + x + 2 \equiv (x+1)(Ax^3 + Bx^2 + Cx + D) + R,$$

so

$$x^4 + x + 2 \equiv Ax^4 + (A + B)x^3 + (B + C)x^2 + (C + D)x + D + R.$$

Equating coefficients of x^4: $1 = A$.

Equating coefficients of x^3: $0 = A + B$, so $B - A$, giving $B = -1$.

Equating coefficients of x^2: $0 = B + C$, so $C = -B$, giving $C = 1$.

Equating coefficients of x: $1 = C + D$, so $D = 1 - C$, giving $D = 0$.

Equating coefficients of x^0: $2 = D + R$, so $R = 2 - D$, giving $R = 2$.

The quotient is $x^3 - x^2 + x$ and the remainder is 2.

Long division approach

To divide $x^4 + x + 2$ by $x + 1$, you start by laying out the division as with integers. It is wise to indicate the absence of any x^3 or x^2 terms by writing $0x^3$ and $0x^2$ as placeholders, just as you use zeros to indicate the absence of any hundreds or tens in the number 4007. So you have:

$$x + 1\overline{)\,x^4 + 0x^3 + 0x^2 + x + 2}$$

Now note that the x (the leading term of $x + 1$) divides into the leading term of $x^4 + x + 2$, namely x^4, exactly x^3 times, so you get:

$$\begin{array}{rl} & x^3 \\ x + 1 & \overline{)\,x^4 + 0x^3 + 0x^2 + x + 2} \\ & \underline{x^4 + x^3} \\ & -x^3 \end{array}$$

You continue the division in this way: bringing down the next term, $0x^2$, and noting that x divides into $-x^3$ exactly $-x^2$ times, you get:

$$\begin{array}{rl} & x^3 - x^2 \\ x + 1 & \overline{)\,x^4 + 0x^3 + 0x^2 + x + 2} \\ & \underline{x^4 + x^3} \\ & -x^3 + 0x^2 \\ & \underline{-x^3 - x^2} \\ & x^2 \end{array}$$

Continuing in this way, the final result is

$$
\begin{array}{r|l}
 & x^3 - x^2 + x \\
\hline
x+1 & x^4 + 0x^3 + 0x^2 + x + 2 \\
 & \underline{x^4 + x^3} \\
 & \quad -x^3 + 0x^2 \\
 & \quad \underline{-x^3 - x^2} \\
 & \qquad\quad x^2 + x \\
 & \qquad\quad \underline{x^2 + x} \\
 & \qquad\quad (0x) + 2
\end{array}
$$

So the quotient is $x^3 - x^2 + x$ and the remainder is 2.

EXAMPLE 1.4.2

Find the quotient and remainder when $x^4 + 3x^2 - 2$ is divided by $x^2 - 2x + 2$.

Equating coefficients approach

The result in the box states that the degree of the remainder is less than 2, so assume that it is a linear polynomial. Let the quotient be $Ax^2 + Bx + C$, and the remainder be $Rx + S$. Then

$$x^4 + 3x^2 - 2 \equiv (x^2 - 2x + 2)(Ax^2 + Bx + C) + Rx + S,$$

so $$x^4 + 3x^2 - 2 \equiv Ax^4 + (-2A + B)x^3 + (2A - 2B + C)x^2 + (2B - 2C + R)x + 2C + S.$$

Equating coefficients of x^4: $1 = A$.

Equating coefficients of x^3: $0 = -2A + B$, so $B = 2A$, giving $B = 2$.

Equating coefficients of x^2: $3 = 2A - 2B + C$, so $C = 3 - 2A + 2B$, giving $C = 5$.

Equating coefficients of x: $0 = 2B - 2C + R$, so $R = -2B + 2C$, giving $R = 6$.

Equating coefficients of x^0: $-2 = 2C + S$, so $S = -2 - 2C$, giving $S = -12$.

The quotient is $x^2 + 2x + 5$ and the remainder is $6x - 12$.

Long division approach

When dividing $x^4 + 3x^2 - 2$ by $x^2 - 2x + 2$, the leading term of $x^2 - 2x + 2$ is x^2, so the calculation goes as follows:

$$
\begin{array}{r|l}
 & x^2 + 2x + 5 \\
\hline
x^2 - 2x + 2 & x^4 + 0x^3 + 3x^2 + 0x - 2 \\
 & \underline{x^4 - 2x^3 + 2x^2} \\
 & \quad 2x^3 + x^2 + 0x \\
 & \quad \underline{2x^3 - 4x^2 + 4x} \\
 & \qquad\quad 5x^2 - 4x - 2 \\
 & \qquad\quad \underline{5x^2 - 10x + 10} \\
 & \qquad\qquad\quad 6x - 12
\end{array}
$$

So again, it is found that the quotient is $x^2 + 2x + 5$ and the remainder is $6x - 12$.

Cambridge University Press
978-1-316-60022-1 — Cambridge International AS and A Level Mathematics: Pure Mathematics 2 and 3 Revised Edition Coursebook
Hugh Neill , Douglas Quadling , Julian Gilbey
Excerpt
More Information

When you are dividing by a *linear* polynomial, there is a quick way of finding the remainder. For example, in Example 1.4.1, when $x^4 + x + 2$ was divided by $x + 1$, the first line of the solution was:

$$x^4 + x + 2 \equiv (x + 1)(Ax^3 + Bx^2 + Cx + D) + R.$$

Since this is an identity, it is true for all values of x and, in particular, it is true for $x = -1$. Putting $x = -1$ in the left side, you get $(-1)^4 + (-1) + 2 = 2$; putting $x = -1$ in the right side, you get $0 \times (A(-1)^3 + B(-1)^2 + C(-1) + D) + R$, which is simply R. Therefore $R = 2$.

Similar reasoning leads to the remainder theorem.

Remainder theorem

When a polynomial p(x) is divided by $x - t$,

the remainder is the constant p(t).

Proof When $p(x)$ is divided by $x - t$, let the quotient be $q(x)$ and the remainder be R. Then

$$p(x) \equiv (x - t)q(x) + R.$$

Putting $x = t$ in this identity gives $p(t) = 0 \times q(t) + R = R$, so $R = p(t)$.

EXAMPLE 1.4.3

Find the remainder when $x^3 - 3x + 4$ is divided by $x + 3$.

Let $p(x) = x^3 - 3x + 4$. Then $p(-3) = (-3)^3 - 3 \times (-3) + 4 = -27 + 9 + 4 = -14$. By the remainder theorem, the remainder is -14.

EXAMPLE 1.4.4

When the polynomial $p(x) \equiv x^3 - 3x^2 + ax + b$ is divided by $x - 1$ the remainder is -4. When $p(x)$ is divided by $x - 2$ the remainder is also -4. Find the remainder when $p(x)$ is divided by $x - 3$.

By the remainder theorem, when $p(x)$ is divided by $x - 1$, the remainder is $p(1) = 1^3 - 3 \times 1^2 + a + b = a + b - 2$. Therefore $a + b - 2 = -4$, so $a + b = -2$.

Similarly, $p(2) = 2^3 - 3 \times 2^2 + 2a + b = 2a + b - 4$, so $2a + b - 4 = -4$ and $2a + b = 0$.

Solving the equations $a + b = -2$ and $2a + b = 0$ simultaneously gives $a = 2$ and $b = -4$, making the polynomial $p(x) = x^3 - 3x^2 + 2x - 4$.

The remainder on division by $x - 3$ is $p(3) = 3^3 - 3 \times 3^2 + 2 \times 3 - 4 = 2$.

The remainder theorem is useful for finding the remainder when you divide a polynomial by a linear polynomial such as $x - 2$, but it doesn't tell you how to find the remainder when you divide by a linear polynomial such as $3x - 2$. To do this, you need the extended form of the remainder theorem.

