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978-1-316-51373-6 — Introductory Incompressible Fluid Mechanics
Frank H. Berkshire , Simon J. A. Malham , J. Trevor Stuart
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Introductory Incompressible Fluid Mechanics

This introduction to the mathematics of incompressible fluid mechanics and its applications keeps prerequisites to a minimum – only a background knowledge in multivariable calculus and differential equations is required. Part I covers inviscid fluid mechanics, guiding readers from the very basics of how to represent fluid flows through to the incompressible Euler equations and many real-world applications. Part II covers viscous fluid mechanics, from the stress/rate of strain relation to deriving the incompressible Navier–Stokes equations, through to Beltrami flows, the Reynolds number, Stokes flows, lubrication theory and boundary layers. Also included is a self-contained guide on the global existence of solutions to the incompressible Navier–Stokes equations. Students can test their understanding on 100 progressively structured exercises and look beyond the scope of the text with carefully selected mini-projects. Based on the authors' extensive teaching experience, this is a valuable resource for undergraduate and graduate students across mathematics, science and engineering.

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Introductory Incompressible Fluid Mechanics

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To Rosie and Duncan, Charlie, Chloe and Emily

To Tamsin, Alexander, William and Amelia

To the late Christine Mary Stuart and Andrew, David and Katherine

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Contents

Preface

The goal of this textbook is to introduce the mathematical theory of incompressible fluid mechanics and its applications. It grew out of lectures for a new course on the subject given by SJAM to final-year undergraduate as well as graduate students in autumn 2010. The basic material for the lectures was drawn from lecture notes on 'Ideal Fluid Mechanics' devised by Frank and lecture notes on 'Viscous Fluid Mechanics' devised by Trevor. In October 2015, David Tranah from Cambridge University Press suggested these lecture notes might be the seed for a textbook. After some delay, we all met in autumn 2018, and the plans and structure for the book were laid. Once writing was in full swing, the book grew and developed substantially, and slightly transformed its character. However, we hope the character of the book retains/includes the following essential features, it is accessible, comprehensive, mathematical, practical, engaging and useful. Let us briefly expand on these. We attempt to take interested students on a full journey. This journey starts with the very basic notion of how to represent the flow, carries the reader through analytical solutions and/or pragmatic approximations to practical flow scenarios, problems and applications, and ends at/on the issue of the global existence of solutions to the fluid equations themselves. The practical and mathematical go hand in hand, and a central theme of the book is the rigorous pursuit of the underlying mathematics and mathematical equations to obtain analytical solutions for the applied flow scenarios concerned. Another complementary and distinguishing theme is the plethora of extensive exercises we have included. These are styled progressively and aimed to complement the main theory and examples in the respective chapters.

We believe that Part I of the book (Inviscid Flow) is suitable for penultimate-year or final-year undergraduate students, while Part II (Viscous Flow) is suitable for final-year undergraduate and graduate students. This of course depends on mathematical background and knowledge. We include here mathematics, physics, science and engineering students, who are our target audience. We hope such students use this book as an introduction, springboard and long-term reference for their fluid mechanics knowledge and experience.

Many classical and contemporary classical textbooks have been invaluable in our preparation and exposition. Their influence is discernable throughout parts of the book. In particular, in general throughout, we relied on the classical textbooks by Batchelor, Currie, Kundu and Cohen, Lamb, Landau and Lifshitz, Lighthill, Panton, Paterson, Tritton and Van Dyke, as well as some classical, more contemporary textbooks by Childress, Chorin and Marsden, Majda and Bertozzi and Ockendon and Ockendon.

Other classical textbooks were invaluable to specific parts of the book and are cited at those junctures.

We would like to thank Darren Crowdy, Heiko Gimperlein, Robin Knops, Andrew Lacey, Marcel Oliver and Bernd Schroers for reading through the manuscript and their very helpful comments and suggestions. We are particularly grateful to Bernd Schroers for 'road-testing' the original notes and his invaluable suggestions for improvement. We also thank Daniel Coutand, Ioannis Stylianidis and Callum Thompson for their suggestions and input, as well as all the undergraduate/graduate students who pointed out typos along the way. We are also in debt to, and gratefully thank, the referees whose helpful constructive comments and suggestions significantly helped to improve the original raw manuscript. We would also like to thank Rachel Norridge, Anna Scriven and the whole production team at CUP for their invaluable help and the fantastic smooth job they made of the final production stages of the book. Lastly, many thanks to David Tranah of Cambridge University Press for seeing the potential, and then deftly and patiently guiding us through the whole writing and production process.
The Matlab files used to generate Figures 1.1, 1.2, 1.6, 2.6, 2.16, 3.14, 3.15, 3.16, 3.18, 4.4, 4.5, 5.6 and 6.4 are available on request from SJAM.

Introduction

The derivation of the equations of motion for an *ideal fluid* by Euler in 1755, and then for a *viscous fluid* by Navier (in 1822) and Stokes (in 1845), was a *tour-de-force* of eighteenth and nineteenth-century mathematics. These equations have been used to describe and explain so many physical phenomena around us in nature that currently billions of dollars of research grants in mathematics, science and engineering now revolve around them. They can be used to model the coupled atmospheric and ocean flow used by the meteorological office for weather prediction (generally incompressible flow) down to any application in chemical engineering you can think of, say to development of the thrusters on NASA's Apollo programme rockets (generally compressible flow). The incompressible *Navier–Stokes equations* are given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u} - \frac{1}{\rho} \nabla p + \boldsymbol{f},$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is a fluid velocity vector, $p = p(\boldsymbol{x}, t)$ is the pressure and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ is an external force field (per unit mass). The vector $\boldsymbol{x}$ records a position in space and t records the time elapsed. The constants ρ and ν are the mass density and kinematic viscosity, respectively. The frictional force due to stickiness of a fluid is represented by the term $\nu \nabla^2 \boldsymbol{u}$. An inviscid flow corresponds to the case $\nu = 0$, when the equations above are known as the incompressible *Euler equations* for an ideal flow. A viscous flow corresponds to $\nu > 0$ and the incompressible Navier–Stokes equations above. Herein we derive the incompressible Euler and Navier–Stokes equations and in the process learn about the subtleties of fluid mechanics and along the way see lots of interesting applications.

Part I of this book focuses on inviscid flow, while Part II focuses on viscous flow, though the two parts do naturally filter into and interact with each other. Our target audience is expected to have general *background mathematical knowledge* as follows, while further knowledge specific to particular chapters is outlined in the brief overview of the book just below. We expect the reader to have mastered some basic real analysis, multivariable differential and integral calculus, including knowledge of the gradient, divergence and curl operators, as well as path integration, Stokes' Theorem and the Divergence Theorem. Some knowledge of the solution methods for ordinary differential equations as well as basic methods of solution, such as separation of variables, for linear partial differential equations is also expected. Lastly, the chain rule for multivariable functions is so important we record it here. Given a scalar differentiable function $f = f(u, v, w)$ of three variables u, v and w, which are themselves scalar differentiable functions $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$ of three variables x, y and z, then we have:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x},$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y},$$
$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z},$$

with the obvious extension to more variables if required. The matrix version of this identity $\nabla_{\boldsymbol{x}} f = (\nabla_{\boldsymbol{x}} \boldsymbol{u})^{\mathrm{T}} \nabla_{\boldsymbol{u}} f$ is often convenient, where $\boldsymbol{x} = (x, y, z)^{\mathrm{T}}$ and $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$ are vectors with the superscript 'T' denoting transpose, and ∇ represents the gradient operator.

There are seven chapters in this book, three in Part I and four in Part II. A brief *overview* is as follows. In Chapter 1 we discuss how to represent or record the fluid flow in terms of the local velocity field. We also consider the law of conservation of mass which locally, mathematically, is represented by the continuity equation. With this in hand we can quantify what the adjective 'incompressible' means in our title. We further quantify the transport of parcels of fluids and their acceleration. We round off the chapter by introducing the vorticity vector field and the rate of strain matrix. We use Newton's Second Law in Chapter 2, resolving the relevant forces and locally equating them to mass times acceleration, to derive the evolution equation for the fluid velocity field, the Euler equations for an inviscid fluid. The rest of this chapter explores many and varied applications of the Euler equations to real-world fluid flows from the Venturi tube through to channel flow and water waves. We consider a special class of inviscid fluid flow in Chapter 3, one where the flow is two-dimensional and the vorticity vector, which quantifies local fluid rotation, is everywhere zero. In other words, the fluid flow is irrotational. Remarkably, such a flow both applies to aid the basic understanding of dynamic lift of an aerofoil and thus flight, and can be represented by complex variables. Indeed, complex analysis plays a crucial role in the analysis of this important real-world application, and a basic knowledge is assumed. Further subtleties of this application requiring a viscous flow prescription are revisited in Chapter 6, discussed presently. Part II begins with Chapter 4, in which we again apply Newton's Second Law to derive the incompressible Navier–Stokes equations, only this time we include more realistic viscous or diffusive forces. Resolving the viscous force is a much more subtle affair and a large section is devoted to its careful derivation. Some background knowledge on eigenvalues and eigenvectors is assumed. The rest of the chapter is concerned with special classes of incompressible flows that have exact solutions and finishes by introducing the Reynolds number. Chapter 5 looks at flows which are characterised by small Reynolds numbers, which roughly translates to them being either akin to sticky flows, think of treacle or syrup, or flows restricted to a thin layer. The former flows can be approximated by the simpler 'Stokes flow', while the latter flows are relevant to lubrication theory, and together they constitute the two main sections of this chapter. The exercises in this chapter explore the consequences and limits of such approximations. Though there is no need for it to be a prerequisite, some asymptotic analysis is utilised in a few of these exercises. The practical theory of flight is reprised in Chapter 6, where the notion of boundary layers and the equations prescribing them are introduced. Boundary layer and/or flow separation is also discussed, though a full quantitative treatment is beyond the scope of this book. Finally, Chapter 7 covers the basic results concerning the well-posedness and regularity of the three-dimensional incompressible Navier–Stokes equations. A comprehensive analysis of the classical regularity results is presented and/or given as explicit, carefully staged exercises. Some basic knowledge of functional and Fourier analysis is assumed. See below for more details on this chapter. The full journey through all seven chapters spans a large section of applied mathematical knowledge. In practice, for most students and/or courses this will often not be required or expected, and certainly not in a single semester or maybe even two semesters.

A natural associated topic to remark on at this point is that of fluid 'turbulence'. For the sake of brevity we have not included any discussion on this herein, apart from naturally occuring allusions to the topic when we briefly mention the stochastically driven Navier–Stokes equations in Section 4.9(7) and turbulent boundary layers in Chapter 6. Such, possibly chaotic, features are relevant for weather prediction, see Section 2.10(7), and many problems in engineering and physics.

We consider a unique feature of the book to be the breadth and depth of the *exercises* provided at the end of each chapter. There are nearly one hundred pages of exercises. They are an integral part of the learning process and knowledge acquisition intended for the reader. They either consolidate knowledge or extend knowledge given in the main text of the chapter. All the exercises are staged, i.e. they are split into structured progressive parts which combine to establish an overall result or outcome. Each part generally states a partial goal/result to achieve. Thus, if the reader is temporarily stuck completing one part, they can still use the partial result in the following parts to try to establish/complete them. A large proportion of the exercises have been questions on past exam papers and some of the more advanced/extended exercises could certainly be used as components of a take-home exam or project. Indeed, at the end of most chapters we also present some *projects* that involve understanding some particular material of the chapter, completing the relevant exercises and using that as a basis to explore further the topic/concept concerned. At the end of most chapters is also a *notes* section, where further details, applications, extensions, questions or related material are explored. We also use these sections to cover, briefly, specific technical material beyond the scope of the main text in the chapter.

Another unique feature of the book is Chapter 7. The material therein on the regularity of solutions to the three-dimensional incompressible Navier–Stokes equations is not normally included in textbooks at this level. However, the importance of this material and question, especially as it constitutes one of the Millennium Prize Problems, impels us to include it. The goal is to make the material and results accessible, and to be as direct and self-contained as possible. To achieve this we have taken a slightly different approach to the standard one and have chosen to prove the existence of weak solutions via a Fourier-series representation for the solution. This means our setting is countably discrete and we only require more elementary concepts from real analysis, convergence, compactness and so forth. We have endeavoured to cover the classical results on weak and strong solutions as comprehensively, explicitly and succinctly as possible.

Lastly, let us remark on modelling and analytical solution strategies. Our focus herein is on incompressible flow scenarios. Liquid flows are in general incompressible, indeed low-Mach-number gas flows can be considered as such. Another distinguishing issue is whether it is sufficient to model the scenario by the inviscid flow Euler equations, or viscous flow Navier–Stokes equations, or both – such as in boundary-layer theory where in one region viscosity effects are important while in another viscous effects are negligible. The issue we wish to emphasise is this. Given a modelling scenario to which either the incompressible Euler or Navier–Stokes equations apply, the goal is to derive analytical solutions either to these equations themselves or to reduced versions of these equations under some underlying physical or geometric assumptions. Hence, for example, the flow may be particularly sticky or a slow/creeping flow corresponding to a low Reynolds number and to a very good approximation we can reduce the incompressible Navier–Stokes equations to the Stokes equation. Or the flow may take place in an asymptotically thin fluid layer and to a very good approximation we can reduce the incompressible Navier–Stokes equations to the Reynolds lubrication equation, and so forth. In all such cases we should utilise further properties. For example, to an accurate approximation is the flow scenario rotationally invariant about a given axis, i.e. axisymmetric? Is there any swirl? Is the flow steady/stationary? Any such symmetries can be utilised to simplify the model equations further to obtain a more tractable reduced system of differential equations. Once all natural symmetries have been taken into account, can we solve the resulting system of differential equations? To find solutions to such reduced equations it is often useful to make a guess at the general form of the solution flow, i.e. pose a solution ansatz. For example, the boundary conditions may indicate an overall solution form allowing us to reduce the approximate model equations further. Or prescribing such a specific flow ansatz in particular flow regions can also be helpful, for example, is the flow near the axis of rotation in an axisymmetric flow approximately a solid-body flow? In which case we might want to employ that fact to represent the flow in that region and couple it by continuity to the neighbouring flow further away from the axis of symmetry. Having obtained a solution, is it physically realistic? Does it represent what is observed? Does it predict flow behaviour not hitherto seen? For example, Moffatt vortices were predicted before they were observed! How can we utilise the knowledge we have gained to optimise processes, such as energy transmission from a wind turbine, or via the introduction of a control mechanism into the system, pushing the system towards a given desired state? As a simple example, the peloton of riders in the Tour de France will change its shape from a V-shaped headed mass in head winds to different-sized echelons depending on the direction of cross winds, in order to minimise energy expenditure due to air resistance. Besides the practical flow scenarios touched upon in this book, there are many more real-world flows the analysis and equations developed herein can be applied to and we should be on the lookout for. These days there are also many high-quality videos available online of all sorts of fluid-flow phenomena, to add to the classical *An Album of Fluid Motion* by Van Dyke. Chris Hadfield's experiments on the International Space Station also provide an out-of-this-world fluid-flow and surface-tension experience.