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Frank H. Berkshire , Simon J. A. Malham , J. Trevor Stuart
Excerpt
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Part I

Inviscid Flow

1 Flow and Transport

1.1 Fluids and the Continuum Hypothesis

A material exhibits *flow* if shear forces, however small, lead to a deformation which is unbounded – we could use this as a definition of a *fluid*. A *solid* has a fixed shape, or at least a strong limitation on its deformation when force is applied to it. Within the category of 'fluids', we include liquids and gases. The main distinguishing feature between these two fluids is the notion of compressibility. Gases are usually compressible – as we know from everyday aerosols. Liquids are generally incompressible – a feature essential to all modern car brakes. However, some gas flows can also be incompressible, particularly at low speeds.

Fluids can be further subcategorised. There are *ideal* or *inviscid* fluids. In such fluids, the *only* internal force present is pressure, which acts so that fluid flows from a region of high pressure to one of low pressure. The equations for an ideal fluid have been applied to wing and aircraft design (as a limit of high Reynolds-number flow). However, fluids can exhibit internal frictional forces which model a 'stickiness' property of the fluid which involves energy loss – these are known as *viscous* fluids. Some fluids/material known as 'non-Newtonian or complex fluids' exhibit even stranger behaviour, their reaction to deformation may depend on: (i) past history (earlier deformations), for example some paints; (ii) temperature, for example some polymers or glass; (iii) the size of the deformation, for example some plastics or silly putty.

For any *real* fluid there are three natural length scales:

1. $L_{\text{molecular}}$, the molecular scale characterised by the mean-free-path distance of molecules between collisions;
2. L_{fluid}, the medium scale of a fluid parcel, the fluid droplet in the pipe or ocean flow;
3. L_{macro}, the macro-scale which is the scale of the fluid geometry, the scale of the container the fluid is in, whether a beaker or an ocean.

And, of course, we have the asymptotic inequalities

$$L_{\text{molecular}} \ll L_{\text{fluid}} \ll L_{\text{macro}}.$$

Continuum Hypothesis. We will assume that the properties of an elementary volume/parcel of fluid, however small, are the same as for the fluid as a whole – i.e. we suppose that the properties of the fluid at scale L_{fluid} propagate all the way down and through the molecular scale $L_{\text{molecular}}$. This is the *continuum assumption*. For everyday fluid mechanics engineering, this assumption is extremely accurate, see Chorin and Marsden (1990, p. 2).

1.2 Conservation Principles

Our derivation of the basic equations underlying the dynamics of fluids is based on three basic conservation principles:

1. *Conservation of mass*, mass is neither created or destroyed.
2. *Newton's Second Law/balance of momentum*, for a parcel of fluid the rate of change of momentum equals the force applied to it.
3. *Conservation of energy*, energy is neither created nor destroyed.

In turn these principles generate the:

1. *Continuity equation*, which governs how the density of the fluid evolves locally and thus indicates compressibility properties of the fluid.
2. *Navier–Stokes equations* of motion for a fluid, which indicate how the fluid moves around from regions of high pressure to those of low pressure and with the effects of viscosity.
3. *Equation of state*, which indicates the mechanism of energy exchange within the fluid.

1.3 Fluid Prescription

A crucial task is to decide how we wish to represent the fluid flow. We will use the Eulerian prescription as follows. Consider a fluid in a container, what information do we need in order to fully describe the 'state' of the fluid flow? Well, imagine that at every fixed spatial position in the fluid we placed a weather vane that could pivot three-dimensionally, i.e. at every fixed position there is a pointer that points in the direction (three-dimensional) the fluid is flowing at that position. Further, suppose the vane/pointer is also able to record the speed with which the fluid is flowing at that position. If we know the fluid-flow direction and speed at each spatial position, then we know the fluid-flow velocity vector at those positions. Of course that velocity vector could change with time. Thus at any given time the prescription of the velocity vector at every fixed spatial position in the fluid flow, along with some other concomitant fluid-related quantities such as the scalar pressure and mass density given at each spatial position, should be enough to describe the 'state' of the fluid. Given the velocity field at every position and time, we know the fluid flow and we can in principle determine fluid-particle trajectories under that flow. This is the focus of the next section.

1.4 Trajectories and Streamlines

Suppose that our fluid is contained within a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$, where $d = 3$, and we use Cartesian coordinates $\boldsymbol{x} = (x, y, z)^{\mathrm{T}} \in \mathcal{D}$ to label points/positions in $\mathcal{D}$. Imagine a small fluid particle or a speck of dust moving in a fluid flow field prescribed by the *velocity field* $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, where $\boldsymbol{u}$ has $d = 3$ components as follows: $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$. Suppose the position of the particle at time t is recorded by the variables $\boldsymbol{x}(t) = \big(x(t), y(t), z(t)\big)^{\mathrm{T}}$, i.e. by the vector $\boldsymbol{x}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors in the respective coordinate directions x, y and z. We thus have the following equations for the velocity of the particle at time t at position $\boldsymbol{x}(t) = \big(x(t), y(t), z(t)\big)^{\mathrm{T}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = u\big(x(t), y(t), z(t), t\big),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = v\big(x(t), y(t), z(t), t\big),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = w\big(x(t), y(t), z(t), t\big).$$

Definition 1.1 (Particle path or trajectory) The *particle path* or *trajectory* of a fluid particle is the curve traced out by the particle as time progresses. If the particle starts at position $\boldsymbol{x}_0 = (x_0, y_0, z_0)^{\mathrm{T}}$ then its particle path $\boldsymbol{x} = \boldsymbol{x}(t)$ is the solution to the following system of differential equations (the same as those above but here in shorter vector notation) with initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}\big(\boldsymbol{x}(t), t\big).$$

Definition 1.2 (Streamline) A *streamline* is an integral curve of the velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ for t fixed, i.e. it is a curve $\boldsymbol{x} = \boldsymbol{x}(s)$ parameterised by the variable s, that satisfies the following system of differential equations with t held constant and $\boldsymbol{x}(0) = \boldsymbol{x}_0$ at $s = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(s) = \boldsymbol{u}\big(\boldsymbol{x}(s), t\big).$$

Remark 1.3 (Stationary/steady flows) Flows for which $\partial\boldsymbol{u}/\partial t = \boldsymbol{0}$ are said to be *stationary/steady*. For such flows the velocity field $\boldsymbol{u}$ is time-independent, so $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ only, and trajectories and streamlines coincide.

Example 1.4 (Solid-body rotation flow) Suppose a velocity field $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$ depends on position $\boldsymbol{x}$ only and is given by

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\Omega y \\ \Omega x \\ 0 \end{pmatrix}$$

for some non-zero constant $\Omega \in \mathbb{R}$. The particle path for a particle that starts at $\boldsymbol{x}_0 = (x_0, y_0, z_0)^{\mathrm{T}}$ is the integral curve of the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\Omega y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \Omega x,$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 0,$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$. This is a coupled pair of differential equations as the solution to the last equation is $z(t) = z_0$ for all $t \geqslant 0$. There are several methods for solving the pair of equations, one method is as follows. Differentiating the first equation with respect to t we find

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\Omega\frac{\mathrm{d}y}{\mathrm{d}t} \qquad \Leftrightarrow \qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\Omega^2 x.$$

In other words, we are required to solve the linear second-order differential equation for $x = x(t)$ shown. The general solution is

$$x(t) = A\cos(\Omega t) + B\sin(\Omega t),$$

where A and B are arbitrary constants. We now find $y = y(t)$ by substituting this solution for $x = x(t)$ into the first differential equation above as follows:

$$\begin{aligned} y(t) &= -\frac{1}{\Omega}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -\frac{1}{\Omega}(-\Omega A\sin(\Omega t) + \Omega B\cos(\Omega t)) \\ &= A\sin(\Omega t) - B\cos(\Omega t). \end{aligned}$$

Using that $x(0) = x_0$ and $y(0) = y_0$ we find that $A = x_0$ and $B = -y_0$, so the particle path of the particle that is initially at $\boldsymbol{x}_0 = (x_0, y_0, z_0)^{\mathrm{T}}$ is given by

$$\begin{aligned} x(t) &= x_0\cos(\Omega t) - y_0\sin(\Omega t), \\ y(t) &= x_0\sin(\Omega t) + y_0\cos(\Omega t), \\ z(t) &= z_0. \end{aligned}$$

This particle thus traces out a horizontal circular particle path at height $z = z_0$ of radius $(x_0^2 + y_0^2)^{1/2}$. Since this flow is stationary, streamlines coincide with particle paths for this flow. See Figure 1.1.

Example 1.5 (Two-dimensional oscillating flow) Consider the two-dimensional flow field $\boldsymbol{u} = (u, v)^{\mathrm{T}}$, which depends on the two-dimensional position $\boldsymbol{x} = (x, y)^{\mathrm{T}}$ vector and time $t \geqslant 0$, given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0\cos(kx - \alpha t) \end{pmatrix},$$

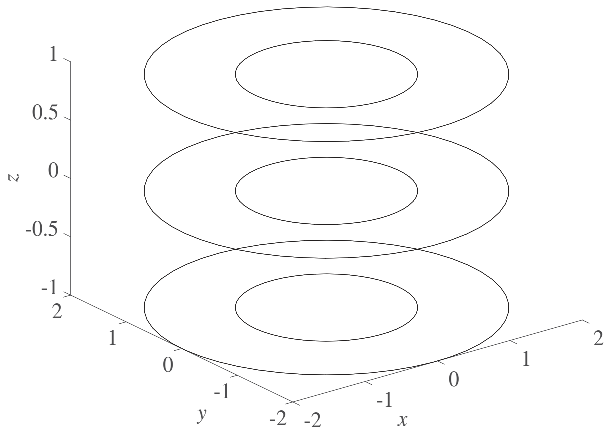

Figure 1.1 Six particle paths corresponding to the solid-body rotation flow in Example 1.4 are shown. Three correspond to the initial positions $x_0 = \sqrt{2}$, $y_0 = \sqrt{2}$ and z_0 equals either -1, 0 or 1. The other three particle paths correspond to $x_0 = \sqrt{2}/2$, $y_0 = \sqrt{2}/2$ and again z_0 equals either -1, 0 or 1. The particles trace out horizontal circular paths, with those tracing out paths, of larger radii travelling faster. All the particles complete one revolution in the same time, hence the nomination as a solid-body rotation flow.

where u_0, v_0, k and α are constants. Let us find the particle path and streamline for the particle at $\boldsymbol{x}_0 = (x_0, y_0)^{\mathrm{T}} = (0,0)^{\mathrm{T}}$ at $t = 0$. Starting with the *particle path*, we are required to solve the coupled pair of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_0,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_0 \cos(kx - \alpha t).$$

We can solve the first differential equation, which tells us

$$x(t) = u_0 t,$$

where we used that $x(0) = 0$. We now substitute this expression for $x = x(t)$ into the second differential equation and integrate with respect to time using $y(0) = 0$ as follows:

$$\begin{aligned} & \frac{\mathrm{d}y}{\mathrm{d}t} = v_0 \cos((ku_0 - \alpha)t) \\ \Leftrightarrow \quad & y(t) = 0 + \int_0^t v_0 \cos((ku_0 - \alpha)\tau)\,\mathrm{d}\tau \\ \Leftrightarrow \quad & y(t) = \frac{v_0}{ku_0 - \alpha} \sin((ku_0 - \alpha)t). \end{aligned}$$

If we eliminate time t between the formulae for $x = x(t)$ and $y = y(t)$ we find that the trajectory through $(0,0)^{\mathrm{T}}$ is

$$y = \frac{v_0}{ku_0 - \alpha} \sin\Big(\Big(k - \frac{\alpha}{u_0}\Big)x\Big).$$

To find the *streamline* through $(0,0)^{\mathrm{T}}$, we fix t and solve the pair of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = u_0,$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = v_0 \cos(kx - \alpha t).$$

As above we can solve the first equation so that $x(s) = u_0 s$ using that $x(0) = 0$. We can substitute this into the second equation and integrate with respect to s, remembering that t is constant, to get

$$\begin{aligned} & \frac{\mathrm{d}y}{\mathrm{d}s} = v_0 \cos(ku_0 s - \alpha t) \\ \Leftrightarrow \quad & y(s) = 0 + \int_0^s v_0 \cos(ku_0 r - \alpha t)\,\mathrm{d}r \\ \Leftrightarrow \quad & y(s) = \frac{v_0}{ku_0}\big(\sin(ku_0 s - \alpha t) - \sin(-\alpha t)\big). \end{aligned}$$

If we eliminate the parameter s between $x = x(s)$ and $y = y(s)$ above, we find the equation for the streamline is

$$y = \frac{v_0}{ku_0}\big(\sin(kx - \alpha t) + \sin(\alpha t)\big).$$

The equation of the streamline through $(0,0)^{\mathrm{T}}$ at time $t = 0$ is thus given by

$$y = \frac{v_0}{ku_0}\sin(kx).$$

As the underlying flow is *not* stationary, as expected, the particle path and streamline through $(0,0)^{\mathrm{T}}$ at time $t = 0$ are distinguished; see Figure 1.2. Finally, let us examine

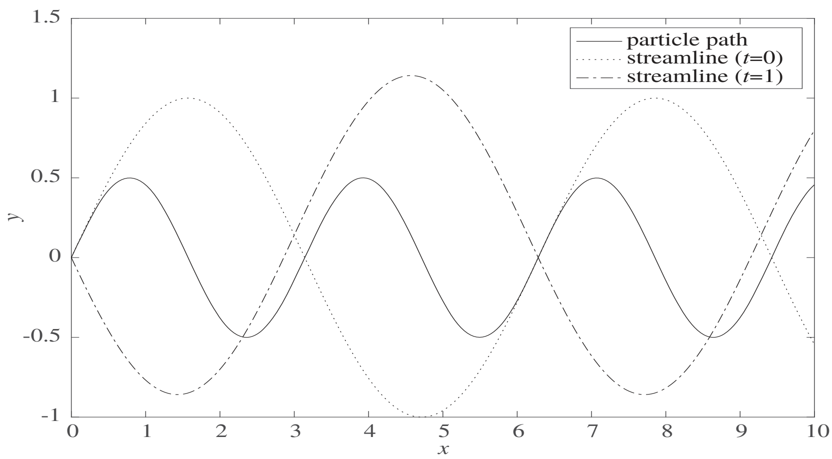

Figure 1.2 For the oscillatory flow in Example 1.5, we plot both the particle path associated with the particle starting from the origin (solid line) and the streamlines through the origin when time is instantaneously frozen, first at $t = 0$ (dotted line) and second at $t = 1$ (dash-dotted line). The other parameters were fixed as $k = 1$, $\alpha = 3$, $u_0 = 1$ and $v_0 = 1$.

two special limits for this flow. As $\alpha \to 0$ the flow becomes stationary and correspondingly the particle path and streamline coincide. As $k \to 0$ the flow is not stationary. In this limit the particle path through $(0,0)^{\mathrm{T}}$ is $y = (v_0/\alpha)\sin(\alpha x/u_0)$, i.e. it is sinusoidal, whereas the streamline is given by $x = u_0 s$ and $y = v_0 s$, which is a straight line through the origin.

Remark 1.6 (Streaklines) A streakline is the locus of all the fluid elements which at some time have passed through a particular point, say $(x_0, y_0, z_0)^{\mathrm{T}}$. We can obtain the equation for a streakline through $(x_0, y_0, z_0)^{\mathrm{T}}$ by solving the ordinary differential equations $(\mathrm{d}/\mathrm{d}t)\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t)$ assuming at $t = t_0$ we have $(x(t_0), y(t_0), z(t_0))^{\mathrm{T}} = (x_0, y_0, z_0)^{\mathrm{T}}$. Eliminating t_0 between the equations generates the streakline corresponding to $(x_0, y_0, z_0)^{\mathrm{T}}$. For example, ink dye injected at the point $(x_0, y_0, z_0)^{\mathrm{T}}$ in the flow will trace out a streakline.

1.5 Continuity Equation

Recall that we suppose our fluid is contained within a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$. Here we assume $d = 3$, but everything we say is true for the collapsed two-dimensional case $d = 2$. Hence $\boldsymbol{x} = (x, y, z)^{\mathrm{T}} \in \mathcal{D}$ is a position/point in $\mathcal{D}$. At each time t we suppose that the fluid has a well-defined *mass density* $\rho = \rho(\boldsymbol{x}, t)$ at the point $\boldsymbol{x}$. Indeed, invoking the continuum hypothesis, at each time t we can compute the mass of fluid inside a small volume centred at $\boldsymbol{x}$, and then consider the ratio of that mass to the volume in the limit as the volume shrinks to zero around the point $\boldsymbol{x}$. The limiting ratio generates the mass density $\rho = \rho(\boldsymbol{x}, t)$. In addition, we note that each fluid particle traces out a well-defined path in the fluid, and its motion along that path is governed by the *velocity field* $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ at time t. Consider an arbitrary fixed subregion $\mathcal{V} \subseteq \mathcal{D}$; see Figure 1.3. The total mass of fluid contained inside the region $\mathcal{V}$ at time t is

$$\int_{\mathcal{V}} \rho(\boldsymbol{x}, t)\, \mathrm{d}V(\boldsymbol{x}),$$

where $\mathrm{d}V = \mathrm{d}V(\boldsymbol{x})$ is the volume measure in $\mathbb{R}^d$. Let us now consider the rate of change of mass inside $\mathcal{V}$. By the principle of conservation of mass, the rate of increase

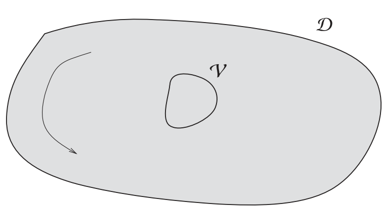

Figure 1.3 The fluid of mass density $\rho(\boldsymbol{x}, t)$ swirls around inside the container $\mathcal{D}$, while $\mathcal{V}$ is an arbitrary fixed subregion.

of the mass in $\mathcal{V}$ is given by the mass of fluid entering/leaving the boundary $\partial\mathcal{V}$ of $\mathcal{V}$ per unit time.

To compute the total mass of fluid entering/leaving the boundary $\partial\mathcal{V}$ per unit time, we consider a small area patch $\mathrm{d}S = \mathrm{d}S(\boldsymbol{x})$ on the boundary of $\partial\mathcal{V}$, which has unit outward normal $\boldsymbol{n}$. The total mass of fluid flowing out of $\mathcal{V}$ through the area patch $\mathrm{d}S = \mathrm{d}S(\boldsymbol{x})$ per unit time is (where '×' is just scalar multiplication)

$$\text{mass density} \times \text{fluid volume leaving per unit time}$$

which is, to leading order,

$$\rho(\boldsymbol{x}, t) \times \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\, \mathrm{d}S(\boldsymbol{x}),$$

where, say, $\boldsymbol{x}$ is at the centre of the area patch $\mathrm{d}S$ on $\partial\mathcal{V}$. Note that to estimate the fluid volume leaving per unit time we have decomposed the fluid velocity at $\boldsymbol{x} \in \partial\mathcal{V}$, time t, into velocity components normal $(\boldsymbol{u} \cdot \boldsymbol{n})$ and tangent to the surface $\partial\mathcal{V}$ at that point. The velocity component tangent to the surface pushes fluid along the surface – no fluid enters or leaves $\mathcal{V}$ via this component. Hence we only retain the normal component – see Figure 1.4.

Returning to the principle of conservation of mass, this is now equivalent to the *integral form of the law of conservation of mass*, which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \rho \,\mathrm{d}V = -\int_{\partial\mathcal{V}} \rho \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}S.$$

That the rate of change of the total mass in $\mathcal{V}$ equals the total rate of change of mass density in $\mathcal{V}$, and the divergence theorem, imply respectively

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \rho \,\mathrm{d}V = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \,\mathrm{d}V \quad \text{and} \quad \int_{\partial\mathcal{V}} (\rho \boldsymbol{u}) \cdot \boldsymbol{n} \,\mathrm{d}S = \int_{\mathcal{V}} \nabla \cdot (\rho \boldsymbol{u}) \,\mathrm{d}V.$$

Using these two relations, the law of conservation of mass is equivalent to

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \,\mathrm{d}V = -\int_{\mathcal{V}} \nabla \cdot (\rho \boldsymbol{u}) \,\mathrm{d}V \quad \Leftrightarrow \quad \int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) \right) \mathrm{d}V = 0.$$

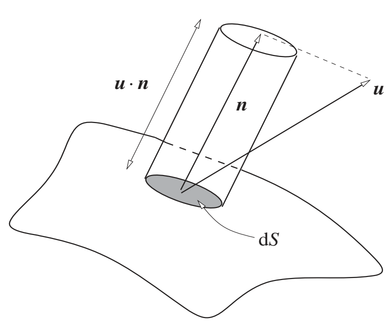

Figure 1.4 The total mass of fluid moving through the patch $\mathrm{d}S$ on the surface $\partial\mathcal{V}$ per unit time is given by the mass density $\rho(\boldsymbol{x}, t)$ times the volume of the cylinder shown, which is $\boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}S$.

We now use that $\mathcal{V}$ is arbitrary to deduce the *differential form of the law of conservation of mass* or *continuity equation* applied pointwise, as follows.

Theorem 1.7 (Continuity equation) *Given a velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, the mass density $\rho = \rho(\boldsymbol{x}, t)$ satisfies the first-order partial differential equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

1.6 Reynolds Transport Theorem

Recall our image of a small fluid particle moving in a prescribed fluid velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$. The velocity of a particle at time t at position $\boldsymbol{x} = \boldsymbol{x}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

As the particle moves in the velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, say from position $\boldsymbol{x} = \boldsymbol{x}(t)$ to a nearby position an instant in time later, two dynamical contributions change: (i) a small instant in time has elapsed and the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, which depends on time, will have changed a little; (ii) the position of the particle has changed in that short time as it moved slightly, and the velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, which depends on position, will be slightly different at the new position. Let us compute the *acceleration* of the particle to observe these two contributions explicitly. By using the chain rule we see that

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(\boldsymbol{x}(t), t) \\
&= \frac{\partial \boldsymbol{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} \\
&= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}\right)\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t} \\
&= (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}.
\end{aligned}$$

Indeed, for any function $F = F(\boldsymbol{x}(t), t)$, scalar- or vector-valued, the chain rule implies

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\boldsymbol{x}(t), t) = \frac{\partial F}{\partial t} + (\boldsymbol{u} \cdot \nabla)F.$$

Definition 1.8 (Material derivative) Given a velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ with components $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$, the partial differential operator $\boldsymbol{u} \cdot \nabla$ is

$$\boldsymbol{u} \cdot \nabla := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

We thus define the *material derivative* following the fluid to be

$$\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla.$$