CHAPTER 1

Dynamic Mechanism Design: Robustness and Endogenous Types

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This article was prepared for an invited session at the 2015 World Congress of the Econometric Society. Through a unifying framework, I survey recent developments in the dynamic mechanism design literature and then introduce two new areas that I expect will draw attention in the years to come: robustness and endogenous types.

1 INTRODUCTION

Long-term contracting plays an important role in a variety of economic problems including trade, employment, regulation, taxation, and finance. Most long-term relationships take place in a "*changing world*," that is, in an environment that evolves (stochastically) over time. Think, for example, of (a) the provision of private and public goods to agents whose valuations evolve over time, as the result of shocks to their preferences or learning and experimentation, (b) the design of multi-period procurement auctions when firms' costs evolve as the result of past investments, (c) the design of optimal tax codes when workers' productivity evolves over time as the result of changes in technology or because of learning-by-doing, (d) the matching of agents whose values and attractiveness is learned gradually over time through private interactions.

Changes to the environment (either due to exogenous shocks, or to the gradual resolution of uncertainty about constant, but unknown, payoffs) are often anticipated at the time of initial contracting, albeit rarely jointly observed by the parties. By implication, optimal long-term contracts must be flexible to

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accommodate such changes, while at the same time provide the parties with incentives to share the information they receive over time.

Understanding the properties of optimal long-term contracts is important both for positive and for normative analysis. It permits one to address questions such as: How does the provision of quantity/quality evolve over time under profit-maximizing contracts? How do the dynamics of the allocations under profit maximization compare to their counterparts under welfare maximization? In particular, when do distortions due to profit maximization decrease over time and vanish in the long run? In what environments does the private observability of the "shocks" (i.e., the changes to the environment subsequent to the signing of the initial contract) play no role? When is the nature of the shocks (i.e., whether they are transitory or permanent) relevant for the dynamics of the decisions under optimal contracts?

The last fifteen years have witnessed significant interest in these questions. Important contributions have been made in extending mechanism design tools to economies in which information evolves over time and a stream of decisions is to be made.[1]

In this article, I first provide a brief overview of the recent dynamic mechanism design literature. I then introduce a simple yet flexible framework that I use in the subsequent sections to review some of the recent contributions. Finally, I discuss two new areas that I expect will attract attention in the near future: robustness and endogenous types.

1.1 Brief Review of the Dynamic Mechanism Design Literature

This section builds on a recent overview that I prepared with Dirk Bergemann for the Journal of Economic Theory Symposium Issue on Dynamic Contracts and Mechanism Design (Bergemann and Pavan, 2015).

An important part of the dynamic mechanism design literature studies how to implement efficient allocations in dynamic settings with evolving private information. The pioneering contributions in this area are Bergemann and Välimäki (2010) and Athey and Segal (2013). The first paper constructs a dynamic pivot transfer scheme under which, in each period, all agents receive their expected marginal flow contribution to social welfare. The scheme guarantees that, in each period, all agents are willing to remain in the mechanism and report truthfully their incremental information, regardless of their beliefs

[1] Mechanism design has been used in static settings to examine a variety of problems including: auctions (Myerson, 1981; Riley and Samuelson, 1981; Cremer and McLean, 1988; Maskin and Riley, 1989); nonlinear pricing (Mussa and Rosen, 1978; Wilson, 1993); bargaining (Myerson and Satterthwaite, 1983; Ausubel and Deneckere, 1989, 1993); regulation (Baron and Myerson, 1982; Laffont and Tirole, 1986); taxation (Mirrlees, 1971); political economy (Dasgupta et al., 1979; Acemoglu et al., 2011); public goods provision (Vickrey, 1961; Clarke, 1971; Groves, 1973; Green and Laffont, 1979); organization design (Cremer, 1995), and voting (Gibbard, 1973, 1977; Satterthwaite, 1975). The reader is referred to Börgers (2015) for an excellent overview of the static mechanism design literature.

about other agents' past and current types (but provided they expect others to report truthfully).[2] The scheme can be thought of as the dynamic analog of the various Vickrey–Clarke–Groves (VCG) schemes proposed in static environments. The paper by Athey and Segal (2013), instead, proposes a transfer scheme under which each agent's "incentives payment," at each period, coincides with the variation in the net present value of the expected externality the agent imposes on other agents, with the variation triggered by the agent's own incremental information. The proposed scheme can thus be thought of as the dynamic analog of the type of schemes proposed by d'Aspremont and Gérard-Varet (AGV) for static settings. Relative to the dynamic pivot mechanism of Bergemann and Välimäki (2010), the Athey and Segal (2013) mechanism has the advantage of guaranteeing budget balance in each period. Contrary to Bergemann and Välimäki (2010), however, it need not guarantee that agents have the incentives to stay in the mechanism in each period.[3]

A second body of work investigates properties of profit-maximizing mechanisms in settings with evolving private information. Earlier contributions include Baron and Besanko (1984), Besanko (1985), and Riordan and Sappington (1987). For more recent contributions, see, among others, Courty and Li (2000), Battaglini (2005), Eső and Szentes (2007), Board (2007), and Kakade et al. (2013).

Pavan, Segal, and Toikka (2014) summarize the above contributions and extend them to a general dynamic contracting setting with a continuum of types, multiple agents, and arbitrary time horizon. The model allows for serial correlation of the agents' information and for the dependence of this information on past allocations. The approach to the design of optimal mechanisms in Pavan, Segal, and Toikka (2014) can be thought of as the dynamic analog of the approach pioneered by Myerson (1981) for static settings, and subsequently extended by Guesnerie and Laffont (1984), Maskin and Riley (1984), and Laffont and Tirole (1986), among others. This approach consists in first identifying necessary conditions for incentive compatibility that can be summarized in an envelope formula for the derivative of each agent's equilibrium payoff with respect to the agent's type. This formula in turn permits one to express transfers as a function of the allocation rule and thereby to express the principal's objective as virtual surplus (i.e., total surplus, net of handicaps that control for the cost to the principal of leaving the agents information rents). The second step then consists in maximizing virtual surplus across all possible allocation rules, including those that need not be incentive compatible. The final step consists in verifying that the allocation rule that solves the relaxed program, along with the transfer rule required by the necessary

[2] The formal solution concept capturing the above properties is periodic ex-post equilibrium.
[3] See also Liu (2014) for an extension of the Bergemann and Välimäki (2010) mechanism to a setting with interdependent valuations.

envelope conditions, constitute a fully incentive-compatible and individually-rational mechanism. This last step typically involves "reverse-engineering," i.e., identifying appropriate primitive conditions guaranteeing that the allocation rule that solves the relaxed program satisfies an appropriate monotonicity condition.

The approach in Pavan, Segal, and Toikka (2014) – reviewed in Section 4, below – adapts the above steps to a dynamic environment. The cornerstone is a dynamic envelope theorem that yields a formula for the evolution of each agent's equilibrium payoff and that must be satisfied in any incentive-compatible mechanism. This formula combines the usual direct effect of a change in the agent's current type on the agent's utility (as in static mechanism design problems) with novel effects stemming from the effect that a change in the current type has on the distribution of the agent's future types. These novel effects, which are specific to dynamic problems, are summarized by *impulse response functions* that describe how a change in the current type propagates throughout the entire type process. A second contribution of Pavan, Segal, and Toikka (2014) is to show that, in Markov environments, the aforementioned dynamic envelope formula, combined with an appropriate *integral monotonicity condition* on the allocation rule, provides a complete characterization of incentive compatibility. The integral monotonicity condition is the dynamic analog of the monotonicity conditions identified in static problems with unidimensional private information but multidimensional decisions (see, among others, Rochet, 1987; Carbajal and Ely, 2013; and Berger et al., 2010). This condition requires that the allocations be monotone in the reported types "on average," where the average is both across time and states, and is weighted by the impulse responses of future types to current ones.

As in static settings, the Myersonian (first-order) approach yields an implementable allocation rule only under fairly stringent conditions. An important question for the dynamic mechanism design literature is thus the extent to which the predictions identified under such an approach extend to environments where global incentive-compatibility constraints bind. This topic is addressed in two recent papers, Garrett and Pavan (2015) and, Garrett, Pavan, and Toikka (2016).[4] These papers do not fully solve for the optimal mechanisms. Instead, they use variational arguments to identify certain properties of the optimal contracts. More precisely, they use perturbations of the allocation policies that preserve incentive compatibility to identify robust properties of the dynamics of the allocations under optimal contracts. I review this alternative variational approach in Section 5, below.[5]

[4] See also Battaglini and Lamba (2015).

[5] The notion of robustness considered in these papers is with respect to the details of the type process. Robustness with respect to the agents' higher-order beliefs is the topic of a by now rich literature well summarized in the monograph by Bergemann and Morris (2012). The type of problems examined in this literature are typically static. For some recent developments to

Another body of the literature studies the design of efficient and profit-maximizing mechanisms in dynamic settings where the agents' private information is static, but where agents or objects arrive stochastically over time. A recent monograph by Gershkov and Moldovanu (2014) summarizes the developments of this literature (see also Bergemann and Said, 2011; Board and Skrzypacz, 2016; Gershkov et al., 2014; and Said, 2011, 2012). Most of the papers in this literature assume that the agents' information is stationary. Instead, Garrett (2016a, 2016b), Hinnosaar (2016), and Ely et al. (2016) combine dynamics originating from stochastic arrivals with dynamics generated by evolving private information. A recent new addition to this literature is Akan et al. (2015); the paper studies a sequential screening environment à la Courty and Li (2000), but in which different agents learn their valuations at different times, with the timing of learning correlated with the agents' initial valuations.[6]

Dynamic mechanism design has also been applied to study optimal insurance, taxation, and redistribution in the so-called "New Dynamic Public Finance" literature. For earlier contributions, see Green (1987), Atkenson and Lucas (1992), and Fernandes and Phelan (2000). For more recent contributions, see Kocherlakota (2005), Albanesi and Sleet (2006), Farhi and Werning (2013), Kapicka (2013a), Stantcheva (2014) and Golosov et al. (2016).

In all the papers above, the evolution of the agents' private information is exogenous. In contrast, the evolution of the agents' information is endogenous in the experimentation model of Bergemann and Välimäki (2010), in the procurement model of Krähmer and Strausz (2011), in the sponsored-search model of Kakade et al. (2013), in the bandit-auction model of Pavan, Segal, and Toikka (2014), in the matching model of Fershtman and Pavan (2016), and in the taxation model of Makris and Pavan (2016). This last paper is reviewed in Section 6, below; it considers a dynamic taxation problem in which the agents' productivity evolves endogenously as the result of learning-by-doing.

Related is also the literature on dynamic managerial compensation. Most of this literature studies optimal compensation schemes in a pure moral hazard setting (see, for example, Prendergast, 2002 for an earlier overview; Sannikov, 2013 for a more recent overview of the continuous-time contracting literature; and the references in Board, 2011 for the subset of this literature focusing on relational contracting). The part of this literature that is most related to the dynamic mechanism design literature is the one that assumes that the manager observes shocks to the cash flows prior to committing his

dynamic environments, see Aghion et al. (2012), Mueller (2015), and Penta (2015). Another strand of the literature studies screening and moral hazard problems in settings in which the principal lacks information about the type distribution, the set of available effort choices, or the technology used by nature to perturb the agent's action. See, for example, Segal (2003), Frankel (2012), Chassang (2013), Garrett (2014), Carroll (2015), and the references therein.

6 See also Krähmer and Strausz (2016) for a discussion of how the analysis of sequential screening in Courty and Li (2000) can be reconducted to a static screening problem with stochastic allocations.

effort (as in the taxation and in the regulation literature); see, for example, Edmans and Gabaix (2011), Edmans et al. (2012), Garrett and Pavan (2012), and Carroll and Meng (2016). This timing is also the one considered in the variational-approach paper by Garrett and Pavan (2015) reviewed in Section 5, below.

Most of the analysis in the dynamic mechanism design literature is in discrete time. One of the earlier papers in continuous time is Williams (2011). For a discussion of the developments of the continuous-time dynamic adverse selection literature and its connection to discrete time, see the recent paper by Bergemann and Strack (2015a) and the references therein.[7]

The dynamic mechanism design literature typically assumes that the designer can commit to her mechanism, with the dynamics of the allocations originating either in evolving private information or in the stochastic arrival and departure of goods and agents over time. A related literature on dynamic contracting under limited commitment investigates the dynamics of allocations in models in which the agents' private information is static but where the principal is unable to commit to future decisions. For earlier contributions to this literature, see, for example, Laffont and Tirole (1988), and Hart and Tirole (1988). For more recent contributions, see Skreta (2006, 2015), Battaglini (2007), Galperti (2015), Maestri (2016), Gerardi and Maestri (2016), Liu et al. (2015), Strulovici (2016), and the references therein. A particular form of limited commitment is considered in Deb and Said (2015). In that paper, the seller can commit to the dynamic contract she offers to each agent, but cannot commit to the contracts she offers to agents arriving in future periods. Partial commitment is also the focus of a recent paper in continuous time by Miao and Zhang (2015), in which both the principal and the agent can walk away from the relationship at any point in time after observing the evolution of the agent's income process.

Another assumption typically maintained in the dynamic mechanism design literature is that transfers can be used to incentivize the agents to report their private information (and/or to exert effort). A few papers investigate dynamic incentives in settings with or without evolving private information, in which transfers are not feasible. An early contribution to this literature is Hylland and Zeckhauser (1979). More recent contributions include Abdulkadiroğlu and Loertscher (2007), Miralles (2012), Kováč et al. (2014), Johnson (2015), Li et al. (2015), Frankel (2016), Johnson (2015), and Guo and Hörner (2016).

Related is also the literature on information design. For a survey of earlier contributions see Bergemann and Välimäki (2006). For more recent developments, including dynamic extensions, see Gershkov and Szentes (2009), Rayo and Segal (2010), Kamenica and Gentzkow (2011), Gentzkow and Kamenica (2015), Bergemann and Morris (2016), Ely et al. (2016), Doval and Ely (2016), Ely, Garrett, and Hinnosaar (2016), and the references

[7] See also Prat and Jovanovic (2014), Strulovici and Szydlowski (2015), and Williams (2015) for recent contributions.

therein. Hörner and Skrzypacz (2016) offer a useful survey of these recent developments. The canonical persuasion model assumes that the designer (the sender) can choose the information structure for the receiver at no cost. In contrast, Calzolari and Pavan (2006a, 2006b), consider models in which a principal first screens the private information of one, or multiple agents, and then passes a garbled version of this information to other agents, or other principals. The design of optimal disclosure rules in screening environments is also the focus of Bergemann and Pesendorfer (2007), Eső and Szentes (2007), Bergemann and Wambach (2015), and Nikandrova and Pancs (2015); all these papers study the design of optimal information structures in auctions.

Finally, dynamic mechanism design is related to the literature on information acquisition in mechanism design (see Bergemann and Välimäki (2002, 2006) and the references therein for earlier contributions, and Gershkov and Szentes (2009), and Krähmer and Strausz (2011) for some recent developments).

2 SIMPLE DYNAMIC SCREENING MODEL

In this section, I introduce a simple dynamic screening model that I use in the next four sections to illustrate some of the key ideas in the dynamic mechanism design literature.

The principal is a seller, the agent is a buyer. Their relationship lasts for $T \in \mathbb{N} \cup \{\infty\}$ periods, where T can be either finite or infinite. Time is discrete and indexed by $t = 1, 2, \ldots, T$. Both the buyer and the seller have time-additively-separable preferences given, respectively, by

$$U^P = \sum_t \delta^{t-1}(p_t - C(q_t)) \quad \text{and} \quad U^A = \sum_t \delta^{t-1}(\theta_t q_t - p_t)$$

where $q_t \in \mathcal{Q} \subset \mathbb{R}$ denotes the quantity exchanged in period t, $\theta_t \in \Theta_t$ denotes the buyer's period-t marginal value for the seller's product, p_t denotes the *total* payment from the buyer to the seller in period t, $\delta \geq 0$ denotes the common discount factor, and $C(q_t)$ denotes the cost to the seller of providing quantity q_t.[8] The function $C(\cdot)$ is strictly increasing, convex, and differentiable.

Let $F \equiv (F_t)$ denote the collection of kernels describing the evolution of the buyer's private information, with F_1 denoting the initial distribution over Θ_1 and, for all $t \geq 2$, $F_t(\cdot \mid \theta_{t-1})$ denoting the cdf of θ_t given θ_{t-1}. Note that the above specification assumes the process is Markov and exogenous.

The sequence of events is the following.

- At $t = 0$, i.e., prior to entering any negotiations with the principal, the buyer privately learns θ_1.

[8] The results for a static relationship can be read from the formulas below for the dynamic environment by setting $\delta = 0$.

- At $t = 1$, the seller offers a mechanism $\varphi = (\mathcal{M}, \phi)$. The latter consists of a collection of mappings

$$\phi_t : \mathcal{M}_1 \times \cdots \times \mathcal{M}_t \to \mathcal{Q} \times \mathbb{R}$$

specifying a quantity–price pair for each possible history of messages $m^t \equiv (m_1, \ldots, m_t) \in \mathcal{M}_1 \times \cdots \times \mathcal{M}_t$, with $\mathcal{M} \equiv (\mathcal{M})_{t=1}^T$ and $\phi \equiv (\phi_t)_{t=1}^T$. A mechanism is thus equivalent to a menu of long-term contracts. If the buyer refuses to participate in φ, the game ends and both players obtain a payoff equal to zero. If the buyer chooses to participate in φ, he sends a message $m_1 \in \mathcal{M}_1$, receives quantity $q_1(m_1)$, pays a transfer $p_1(m_1)$, and the game moves to period 2.
- At the beginning of each period $t \geq 2$, the buyer privately learns θ_t. He then sends a new message $m_t \in \mathcal{M}_t$, receives the quantity $q_t(m^t)$, pays $p_t(m^t)$ to the principal, and the game moves to period $t + 1$.
- $\cdots$
- At $t = T + 1$ the game is over (in case T is finite).

Remark The game described above assumes that the principal (here the seller) perfectly commits to the mechanism φ. It also assumes that at any period $t \geq 2$ the buyer is constrained to stay in the relationship if he signed on in period 1. When the agent has "deep pockets," there are, however, simple ways to distribute the payments over time so that it is in the interest of the buyer to remain in the relationship at all periods, irrespective of what he did in the past.[9] $\|$

The principal's problem consists in designing a mechanism that disciplines the provision of quantity and the payments over time. Because the principal can commit, the Revelation Principle[10] applies and one can without loss of optimality restrict attention to direct mechanisms in which $\mathcal{M}_t = \Theta_t$ all t and such that the agent finds it optimal to report truthfully at all periods. For simplicity, hereafter, I drop the message spaces and identify such a mechanism directly with the policies $\chi = \langle \mathbf{q}, \mathbf{p} \rangle$ that it induces, where, for any $t \geq 1$, $q_t : \Theta^t \to \mathcal{Q}$ is the period-t output policy and $p_t : \Theta^t \to \mathbb{R}$ the payment policy, with $\Theta^t = \Theta_1 \times \cdots \times \Theta_t$.[11] The principal designs χ so as to maximize

$$\mathbb{E}\left[\sum_t \delta^{t-1}(p_t(\theta^t) - C(q_t(\theta^t)))\right]$$

[9] See also Krähmer and Strausz (2015a) for a discussion of interim vs ex-post participation constraints in sequential secreening models.

[10] See, among others, Gibbard (1977), Green and Laffont (1979), Myerson (1979, 1986).

[11] A similar notation will be used hereafter to denote sequences of sets. For example, $\mathcal{A}^t = \mathcal{A}_1 \times \cdots \times \mathcal{A}_t$ with generic element $a^t = (a_1, \ldots, a_t)$.

subject to

$$\mathbb{E}\left[\sum_t \delta^{t-1}(\theta_t q_t(\theta^t) - p_t(\theta^t)) \mid \theta_1\right] \geq 0 \text{ for all } \theta_1 \in \Theta \tag{IR-1}$$

$$\mathbb{E}\left[\sum_{s\geq t} \delta^{s-t}(\theta_s q_s(\theta^s) - p_s(\theta^s)) \mid \theta^t\right]$$
$$\geq \mathbb{E}\left[\sum_{s\geq t} \delta^{s-t}(\theta_s q_s^\sigma(\theta^s) - p_s^\sigma(\theta^s)) \mid \theta^t\right] \text{ for all } t,\ \theta^t \in \Theta^t,\ \text{and } \sigma, \tag{IC-t}$$

where $\boldsymbol{\sigma}$ denotes an arbitrary continuation strategy for the game that starts in period t after the agent has reported truthfully at all previous periods. For any $s \geq t$, any $\theta^s \in \Theta^s$, $q_s^\sigma : \Theta^s \to \mathcal{Q}$ and $p_s^\sigma : \Theta^s \to \mathbb{R}$ are the state-contingent policies induced by the continuation strategy σ under the mechanism $\chi = \langle \mathbf{q}, \mathbf{p} \rangle$. Hereafter, unless otherwise specified, the expectation operator $\mathbb{E}[\cdot]$ is with respect to the entire type sequence $(\theta_s)_{s=1}^T$ under the kernels F.

Note that the above constraints require that the buyer finds it optimal to participate in period 1 and report truthfully "on path," i.e., conditional on having reported truthfully in previous periods. Because the environment is Markov, holding fixed the agent's reports at each period $s < t$, the agent's incentives in period $t \geq 2$ are invariant in the agent's true type θ_s, $s < t$. Hence, the above (IC-t) constraints guarantee that the agent finds it optimal to report truthfully at all histories, not just "on-path."

3 TWO-PERIOD DISCRETE EXAMPLE

To illustrate the trade-offs that determine the dynamics of allocations under optimal mechanisms in the simplest possible way, consider the following environment in which $T = 2$, $\Theta_1 \equiv \{\bar{\theta}, \underline{\theta}\}$, $\underline{\theta} > 0$, $\Delta\theta \equiv \bar{\theta} - \underline{\theta} > 0$, and $\Theta_2 \equiv \{\underline{\theta} - \Delta\theta, \underline{\theta}, \bar{\theta}, \bar{\theta} + \Delta\theta\}$. That is, the buyer has either a high or a low valuation in period 1. In period 2, he then experiences a shock that either raises his valuation by $\Delta\theta$, leaves his valuation unchanged, or reduces his valuation by $\Delta\theta$. To simplify, also assume that the principal's production cost is quadratic, with $C(q) = q^2/2$, all q.

The probability that the buyer is a high type in period 1 (equivalently, the proportion of high types in the cross-section of the population) is $\Pr(\theta_1 = \bar{\theta}) = \nu$. Conditional on θ_1, the transition probabilities are as follows: $\Pr(\bar{\theta}+\Delta\theta|\bar{\theta}) = \bar{x}$, $\Pr(\bar{\theta}|\bar{\theta}) = \bar{\alpha}$, $\Pr(\underline{\theta}|\bar{\theta}) = 1 - \bar{x} - \bar{\alpha}$, $\Pr(\bar{\theta}|\underline{\theta}) = \underline{x}$, $\Pr(\underline{\theta}|\underline{\theta}) = \underline{\alpha}$, and $\Pr(\underline{\theta}-\Delta\theta|\underline{\theta}) = 1-\underline{x}-\underline{\alpha}$. Figure 1 illustrates the situation under consideration.

As usual, the game is solved backwards. Let $U_2^A(\theta_2; \hat{\theta}_1, \hat{\theta}_2) \equiv \theta_2 q_2(\hat{\theta}_1, \hat{\theta}_2) - p_2(\hat{\theta}_1, \hat{\theta}_2)$ denote the agent's period-2 flow payoff when the true period-2 type is θ_2 and the agents reported $(\hat{\theta}_1, \hat{\theta}_2)$ in the two periods. Note

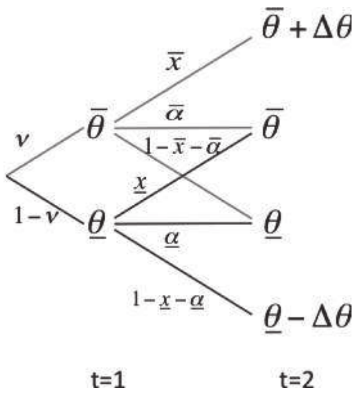

Figure 1 Evolution of Agent's Type.

that the flow period-2 payoff $U_2^A(\theta_2;\hat{\theta}_1,\hat{\theta}_2)$ does not depend on the agent's true period-1 type, θ_1.

Next, let $V_2^A(\theta_1,\theta_2)\equiv U_2^A(\theta_2;\theta_1,\theta_2)$ denote the agent's period-2 flow payoff under truthful reporting (because it is irrelevant whether or not the period-1 report coincides with the true period-1 type, I am replacing $\hat{\theta}_1$ with θ_1 to facilitate the notation).

Incentive compatibility in period 2 then requires that, for all $\theta_1\in\Theta_1$, all $\theta_2,\hat{\theta}_2\in\Theta_2$,

$$V_2^A(\theta_1,\theta_2)\geq U_2^A(\theta_2;\theta_1,\hat{\theta}_2).$$

Because the flow payoffs $\theta_t q_t - p_t$ satisfy the increasing differences property, it is well known, from static mechanism design, that incentive compatibility in period 2 requires that, for all $\theta_1\in\Theta_1$, the output schedules $q_2(\theta_1,\cdot)$ be nondecreasing in θ_2 and the payments $p_2(\theta_1,\cdot)$ satisfy the following conditions:

$$\Delta\theta q_2\left(\theta_1,\bar{\theta}\right)\leq V_2^A(\theta_1,\bar{\theta}+\Delta\theta)-V_2^A(\theta_1,\bar{\theta})\leq\Delta\theta q_2\left(\theta_1,\bar{\theta}+\Delta\theta\right) \tag{1}$$

$$\Delta\theta q_2\left(\theta_1,\underline{\theta}\right)\leq V_2^A(\theta_1,\bar{\theta})-V_2^A(\theta_1,\underline{\theta})\leq\Delta\theta q_2\left(\theta_1,\bar{\theta}\right) \tag{2}$$

$$\Delta\theta q_2\left(\theta_1,\underline{\theta}-\Delta\theta\right)\leq V_2^A(\theta_1,\underline{\theta})-V_2^A(\theta_1,\underline{\theta}-\Delta\theta)\leq\Delta\theta q_2\left(\theta_1,\underline{\theta}\right), \tag{3}$$

both for $\theta_1=\bar{\theta}$ and for $\theta_1=\underline{\theta}$. Along with the monotonicity of the output schedules $q_2(\theta_1,\cdot)$, the above constraints are not only necessary but also sufficient for period-2 incentive compatibility.

Next, let

$$U_1^A(\theta_1;\hat{\theta}_1)\equiv\theta_1 q_1(\hat{\theta}_1)-p_1(\hat{\theta}_1)+\delta\mathbb{E}\left[V_2^A(\hat{\theta}_1,\theta_2)\mid\theta_1\right]$$

denote the payoff that a buyer with initial type θ_1 expects from reporting $\hat{\theta}_1$ in period 1 and, then, reporting truthfully at $t=2$. Observe that the same