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Comparing Dualities in the $K(n)$-local Category

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Abstract

In their work on the period map and the dualizing sheaf for Lubin-Tate space, Gross and the second author wrote down an equivalence between the Spanier-Whitehead and Brown-Comenetz duals of certain type n-complexes in the $K(n)$-local category at large primes. In the culture of the time, these results were accessible to educated readers, but this seems no longer to be the case; therefore, in this note we give the details. Because we are at large primes, the key result is algebraic: in the Picard group of Lubin-Tate space, two important invertible sheaves become isomorphic modulo p.

For John Greenlees, the master of duality.

Introduction

Fix a prime p and and an integer $n \geq 0$, and let $K(n)$ denote the nth Morava K-theory at the prime p. If $n \geq 1$, the $K(n)$-local stable homotopy category has two dualities. First, there is $K(n)$-local Spanier-Whitehead duality $D_n(-)$. This behaves very much like Spanier-Whitehead duality in the ordinary stable category: it has good formal properties, but it can be very hard to compute. Second, there is Brown-Comenetz duality $I_n(-)$, which behaves much like a Serre-Grothendieck duality and, in many ways, is much more computable. One of the key features of the

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$K(n)$-local category is that under some circumstances the two dualities are closely related.

Recall that a finite spectrum X is of type n if $K(m)_*X = 0$ for $m < n$. By [22], any type n spectrum has a $v_n^{p^k}$-self map; that is, there is an integer k and map

$$\Sigma^{2p^k(p^n-1)}X \to X$$

which induces multiplication by $v_n^{p^k}$ in $K(n)_*$. In their papers on the period map and the dualizing sheaf for Lubin-Tate space, Gross and the second author [20] wrote down the following result. Suppose X is a type n-spectrum with a $v_n^{p^k}$-self map and suppose further that p times the identity map of X is zero. Then if $2p > \max\{n^2+1, 2n+2\}$ there is an equivalence in the $K(n)$-local category[1]

$$I_nX \simeq \Sigma^{2p^{nk}r(n)+n^2-n}D_nX \tag{1.1}$$

where $r(n) = (p^n-1)/(p-1) = p^{n-1}+\cdots+p+1$. This equivalence gives a conceptual explanation for many of the self-dual patterns apparent in the amazing computations of Shimomura and his coauthors. See, for example, [32], [31], [5], and [26].

The point of this note is to write down a linear narrative with this result at the center. In some sense, there is nothing new here, as the key ideas can be found scattered through the literature, and other authors have obliquely touched on this topic. A rich early example is in §5 of the paper [8] by Devinatz and the second author, and the important paper of Dwyer, Greenlees, and Iyengar [10] embeds many of the ideas here into a far-reaching and beautiful theory. In another sense, however, there is quite a bit to say, as there are any number of key technical ideas we need to access, some of which have not quite made it into print and others buried in ways that make them hard to uncover. In any case, the result is of enough importance that it deserves specific memorialization.

Here is a little more detail. We fix p and n and let $E = E_n$ be Morava E-theory for n and p. This represents a complex oriented cohomology theory with formal group law a universal deformation of the Honda formal group law H_n of height n. See §1 for more details. As always we write

$$E_*X = \pi_*L_{K(n)}(E \wedge X).$$

The E_*-module E_*X is a graded Morava module: it has a continuous

[1] The bound on p is very slightly different than in [20]; see Proposition 1.9.

and twisted action of the Morava stabilizer group $\mathbb{G}_n = \mathrm{Aut}(H_n, \mathbb{F}_{p^n})$. See Remark 1.5.

There are two key steps to the equivalence (1.1). We have a $K(n)$-local equivalence $I_nX \simeq I_n \wedge D_nX$ where $I_n = I_n(S^0)$; thus, the first step is the identification of the homotopy type of I_n, at least for p large with respect to n. This is also due to Gross and the second author, with details laid out in [33]. The key fact is that I_n is dualizable in the $K(n)$-local category; by [21] this is equivalent to the statement that E_*I_n is an invertible graded Morava module and, indeed, the main result of [33] (interpreting [20]) is that there is an isomorphism of Morava modules

$$E_*I_n \cong E_*S^{n^2-n}[\det]$$

where $S^0[\det] = S[\det]$ is a determinant twisted sphere in the $K(n)$-local category; see Remark 1.26. The number $r(n)$ in (1.1) is an artifact of the determinant; see (1.3.1).

The second key step is an analysis of the $K(n)$-local Picard group $\mathrm{Pic}_{K(n)}$ of equivalence classes of invertible objects in the $K(n)$-local category. As mentioned, we know that a $K(n)$-local spectrum X is invertible if and only if E_*X is an invertible graded Morava module. We also know that the group of invertible graded Morava modules concentrated in even degrees is isomorphic to the continuous cohomology group $H^1(\mathbb{G}_n, E_0^\times)$, where $E_0^\times$ is the group of units in the ring E_0. Hence, if we write $\mathrm{Pic}^0_{K(n)} \subseteq \mathrm{Pic}_{K(n)}$ for the subgroup of objects X with E_*X in even degrees, we get a map

$$e : \mathrm{Pic}^0_{K(n)} \longrightarrow H^1(\mathbb{G}_n, E_0^\times).$$

The map is an injection under the hypothesis $2p > \max\{n^2+1, 2n+2\}$. See Proposition 1.9. This is the origin for the hypothesis on p and n in the equivalence of (1.1): it reduces that equivalence to an algebraic calculation.

It is an observation of [21] that the map $\mathbb{Z} \to \mathrm{Pic}^0_{K(n)}$ sending k to S^{2k} extends to an inclusion of the completion of the integers

$$\mathfrak{Z}_\mathfrak{n} \overset{\text{def}}{=} \lim_k \mathbb{Z}/(p^k(p^n-1)) \to \mathrm{Pic}^0_{K(n)};$$

that is, for any $a \in \mathfrak{Z}_\mathfrak{n}$ we have a sphere S^{2a}. (The phrase "p-adic sphere" is common here, but misleading: $\mathfrak{Z}_\mathfrak{n}$ is not the p-adic integers. See Remark 1.23.) Now let $\lambda = \lim_k\ p^{nk}r(n) \in \mathfrak{Z}_\mathfrak{n}$. The key algebraic result can now be deduced from Proposition 1.30 below: under the composition

$$\mathrm{Pic}^0_{K(n)} \xrightarrow{e} H^1(\mathbb{G}_n, E_0^\times) \to H^1(\mathbb{G}_n, (E_0/p)^\times)$$

the spectra $S[\det]$ and $S^{2\lambda}$ map to the same element. The equivalence (1.1) follows once we observe that if X is type n and has a $v_n^{p^k}$-self map, then there is $K(n)$-local equivalence

$$S^{2\lambda} \wedge X \simeq \Sigma^{2p^{nk}r(n)} X.$$

See Theorems 1.42 and 1.43.

It is worth emphasizing that the algebraic result Proposition 1.30 only requires $p > 2$; it is the topological applications which require the more stringent restrictions on the prime. In fact, the equivalence of dualities in (1.1) can be false if the prime is small. See Remark 1.45.

The plan of this note is as follows: in the first section we give some homotopy theoretic and algebraic background, in the second section we give a discussion of the Picard group, lingering long enough to give details of the structure of $\mathrm{Pic}^0_{K(n)}$ as a profinite $\mathfrak{Z}_\mathfrak{n}$-module. See Proposition 1.18. In Section 3 we discuss the determinant and prove the key Proposition 1.30. In Section 4 we give some discussion of how Spanier-Whitehead and Brown-Comenetz duality behave in the Adams-Novikov Spectral Sequence. In the final section, we give the homotopy theoretic applications.

Acknowledgements

This project began as an attempt to find a conceptual computation of the self-dual patterns apparent in the Shimomura-Yabe calculation [32] of $\pi_* L_{K(2)} V(0)$ at $p > 3$. Then in a conversation with Tobias Barthel it emerged that there was no straightforward argument in print to prove the equivalence of dualities in (1.1). Later, as Guchuan Li was working on his thesis [27] it became apparent that he needed these results and, more, there were constructions once present in the general culture that were no longer easily accessible. Thus a sequence of notes, begun at MSRI in Spring of 2014, have evolved into this chapter. Many thanks for Agnès Beaudry and Vesna Stojanoska for reading through an early draft. Others have surely written down proofs for themselves as well; for example, Hans-Werner Henn once remarked that "the determinant essentially disappears mod p," which is a very succinct summary of Proposition 1.30.

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1.1 Some background

In this section we gather together the basic material used in later sections. All of this is thoroughly covered in the literature and collected here only for narrative continuity.

1.1.1 The $K(n)$-local category

For an in-depth study of the technicalities in the $K(n)$-local category, see Hovey and Strickland [24]. Other introductions can be found in almost any paper on chromatic homotopy theory. We were especially thorough in [3] §2.

Fix a prime p and an integer $n > 0$. In order to be definite we define the nth Morava K-theory $K(n)$ to be the 2-periodic complex oriented cohomology theory with coefficients $K(n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$ with u in degree -2. The associated formal group law over $K(n)_0 = \mathbb{F}_{p^n}$ is the unique p-typical formal group law H_n with p-series $[p]_{H_n}(x) = x^{p^n}$. This is, of course, the nth Honda formal group law. For H_n we have

$$v_n = u^{1-p^n} \in K(n)_{2(p^n-1)}.$$

The $K(n)$-local category is the category of $K(n)$-local spectra.

We also have $K(0) = H\mathbb{Q}$, the rational Eilenberg-MacLane spectrum, and $K(0)$-local spectra are the subject of rational stable homotopy theory.

We define $\mathbb{G}_n = \mathrm{Aut}(H_n, \mathbb{F}_{p^n})$ to be the group of automorphisms of the pair $(H_n, \mathbb{F}_{p^n})$. Since H_n is defined over $\mathbb{F}_p$, there is a splitting

$$\mathrm{Aut}(H_n, \mathbb{F}_{p^n}) \cong \mathrm{Aut}(H_n/\mathbb{F}_{p^n}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

where the normal subgroup is the isomorphisms of H_n as a formal group law over $\mathbb{F}_{p^n}$. We write $\mathbb{S}_n = \mathrm{Aut}(H_n/\mathbb{F}_{p^n})$ for this subgroup.

To get a Landweber exact homology theory which captures more than Morava K-theory, we use the Morava (or Lubin-Tate) theory $E = E_n$. This theory has coefficients

$$E_* = \mathbb{W}[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$$

where again u is in degree -2 but the power series ring is in degree 0. The ring $\mathbb{W} = W(\mathbb{F}_{p^n})$ is the Witt vectors of $\mathbb{F}_{p^n}$.

Note that E_0 is a complete local ring with maximal ideal $\mathfrak{m}$ generated by the regular sequence $\{p, u_1, \ldots, u_{n-1}\}$. We choose the formal group law G_n over E_0 to be the unique p-typical formal group law with p-series

$$[p]_{G_n}(x) = px +_{G_n} u_1 x^p +_{G_n} \cdots +_{G_n} u_{n-1} x^{p^{n-1}} +_{G_n} x^{p^n}. \qquad (1.1.1)$$

Thus $v_i = u_i u^{1-p^i}$, $1 \leq i \leq n-1$, $v_n = u^{1-p^n}$ and $v_i = 0$ if $i > n$. Note that G_n reduces to H_n modulo $\mathfrak{m}$.

We define $E_*X = (E_n)_*X$ by

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

While not quite a homology theory, as it does not take wedges to sums, it is by far our most sensitive algebraic invariant in $K(n)$-local homotopy theory. The group $\mathbb{G}_n$ acts continuously on E_*X making E_*X into a *Morava module*. We will be more precise on this notion below in Remark 1.5.

A basic computation gives

$$E_0E = \pi_0 L_{K(n)}(E \wedge E) \cong \mathrm{map}^c(\mathbb{G}_n, E_0)$$

where map^c denotes the continuous maps. See Lemma 10 of [33] for a proof. The $K(n)$-local E_n-based Adams-Novikov Spectral Sequence now reads

$$H^s(\mathbb{G}_n, E_tX) \Longrightarrow \pi_{t-s} L_{K(n)}X. \qquad (1.1.2)$$

Cohomology here is continuous cohomology.

Remark 1.1 (Lubin-Tate theory) The pair (G_n, E_0) has an important universal property which is useful for understanding the action of $\mathbb{G}_n$.

Consider a complete local ring $(S, \mathfrak{m}_S)$ with $S/\mathfrak{m}_S$ of characteristic p. Define the groupoid of deformations $\mathrm{Def}_{H_n}(S)$ to be the category with objects (i, G) where $i : \mathbb{F}_{p^n} \to S/\mathfrak{m}_S$ is a morphism of fields and G is a formal group law over S with $q_*G = i_*H_n$. Here $q : S \to S/\mathfrak{m}_S$ is the quotient map. There are no morphisms $\psi : (i, G) \to (j, H)$ if $i \neq j$ and a morphism $(i, G) \to (i, H)$ is an isomorphism of formal groups laws $\psi : G \to H$ so that $q_*\psi$ is the identity. These are the $\star$-isomorphisms. By a theorem of Lubin and Tate [28] we know that if two deformations are $\star$-isomorphic, then there is a unique $\star$-isomorphism between them.

Put another way, the groupoid $\mathrm{Def}_{H_n}(S)$ is discrete. Furthermore, E_0 represents the functor of $\star$-isomorphism classes of deformations:

$$\mathrm{Hom}^c_{\mathbb{W}}(E_0, S) \cong \pi_0 \mathrm{Def}_{H_n}(S).$$

Here $\mathrm{Hom}^c_{\mathbb{W}}$ is the set of continuous $\mathbb{W}$-algebra maps. As a universal deformation we can and do choose the formal group law G_n over E_0 to be the p-typical formal group law defined above in (1.1.1).

Remark 1.2 (**The action of the Morava stabilizer group**) We use Lubin-Tate theory to get an action of $\mathbb{G}_n$ on E_0. This exposition follows [18] §3.

Let $g = g(x) \in \mathbb{F}_{p^n}[[x]]$ be an element in $\mathbb{S}_n$. Choose any lift of $g(x)$ to $h(x) \in E_0[[x]]$ and let G_h be the unique formal group law over E_0 so that

$$h : G_h \to G_n$$

is an isomorphism. Since $g : H_n \to H_n$ is an isomorphism over $\mathbb{F}_{p^n}$, the pair (id, G_h) is a deformation of H_n. Hence there is a unique $\mathbb{W}$-algebra map $\phi = \phi_g : E_0 \to E_0$ and a unique $\star$-isomorphism $f : \phi_* G_n \to G_h$. Let ψ_g be the composition

$$\begin{tikzcd} \phi_* G_n \arrow[r, "f"] \arrow[rr, bend right, "\psi_g"'] & G_h \arrow[r, "h"] & G_n \end{tikzcd} \quad . \tag{1.1.3}$$

Note that while G_h depends on choices, the map ϕ_g and the isomorphism ψ_g do not. The map $\mathbb{S}_n \to \mathrm{Aut}(E_0)$ sending g to ϕ_g defines the action of $\mathbb{S}_n$ on E_0. The Galois action on $\mathbb{W} \subseteq E_0$ extends this to an action of all of $\mathbb{G}_n$ on E_0. The action can be extended to all of E_* be noting that $E_2 \cong \tilde{E}^0 S^2 \cong \tilde{E}^0 \mathbb{C}\mathrm{P}^1$ is isomorphic to the module of invariant differentials on the universal deformation G_n. See (1.1.4) below for an explicit formula.

Remark 1.3 (**Formulas for the action**) We make the action of $\mathbb{S}_n$ a bit more precise. By (1.1.3) we have an isomorphism $\psi_g : \phi_* G_n \to G_n$ of p-typical formal group laws over E_0. This can be written

$$\psi_g(x) = t_0(g) +_{G_n} t_1(g) x^p +_{G_n} t_2(g) x^{p^2} +_{G_n} \cdots .$$

This formula defines continuous functions $t_i : \mathbb{S}_n \to E_0$. As in Section 4.1 of [18] we have

$$g_* u = t_0(g) u. \tag{1.1.4}$$

The function t_0 is a crossed homomorphism $t_0 : \mathbb{S}_n \to E_0^\times$; that is,

$$t_0(gh) = [gt_0(h)]t_0(g).$$

Since the Honda formal group is defined over $\mathbb{F}_p$ we can choose the class u to be invariant under the action of the Galois group; hence t_0 extends to crossed homomorphism $t_0 : \mathbb{G}_n \to E_0^\times$ sending $(g,\phi) \in \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{G}_n$ to $t_0(g)$.

Remark 1.4 We record here some basic useful facts about the $K(n)$-local Adams-Novikov Spectral Sequence (1.1.2) which we will use later.

The first two statements are standard and are proved using the action of the center of $Z(\mathbb{G}_n) \subseteq \mathbb{G}_n$ on $E_* = E_*S^0$. There is an isomorphism $\mathbb{Z}_p^\times \cong Z(\mathbb{G}_n)$ sendings $a \in \mathbb{Z}_p^\times$ to the a-series $[a]_{H_n}(x)$ of the Honda formal group. The action of $Z(\mathbb{G}_n)$ on E_0 is trivial and the action on E_* is then determined by the fact that $t_0(a) = a$; that is, a acts on $u \in E_{-2}$ by multiplication by a.

1.) **Sparseness:** If $t \not\equiv 0$ modulo $2(p-1)$, then $H^*(\mathbb{G}_n, E_t) = 0$. If $p = 2$ this is not new information. If $p > 2$ let $C \subseteq Z(\mathbb{G}_n)$ be the cyclic subgroup of Teichmüller lifts of $\mathbb{F}_p^\times$. Then $E_t^C = 0$ and hence

$$H^*(\mathbb{G}_n, E_t) \cong H^*(\mathbb{G}_n/C, E_t^C) = 0.$$

2.) **Bounded torsion:** Suppose $p > 2$ and suppose

$$2t = 2p^k m(p-1) \neq 0$$

with m not divisible by p. Then we have

$$p^{k+1} H^*(\mathbb{G}_n, E_{2t}) = 0.$$

If $p = 2$ write $2t = 2^k(2m+1)$. Then we have

$$2H^*(\mathbb{G}_n, E_{2t}) = 0 \qquad \text{if } k = 1,$$

and

$$2^{k+1} H^*(\mathbb{G}_n, E_{2t}) = 0 \qquad \text{if } k > 1.$$

To get these bounds, first suppose $p > 2$. Let $K = 1 + p\mathbb{Z}_p \subseteq Z(\mathbb{G}_n)$ be the torsion-free subgroup and let $x \in K$ be a topological generator; for example, $x = 1 + p$. The choice of x defines an isomorphism $\mathbb{Z}_p \cong K$. Thus, there is an exact sequence

$$0 \to H^0(K, E_{2t}) \longrightarrow E_{2t} \xrightarrow{x^k - 1} E_{2t} \longrightarrow H^1(K, E_{2t}) \to 0.$$

Thus we see that $p^{k+1}H^1(K, E_{2t}) = 0$ and $H^q(K, E_{2t}) = 0$ if $q \neq 1$. Now use the Lyndon-Hochschild-Serre Spectral Sequence

$$H^p(\mathbb{G}_n/K, H^q(K, E_{2t})) \Longrightarrow H^{p+q}(\mathbb{G}_n, E_{2t})$$

to deduce the claim. At the prime 2 let $x \in \mathbb{Z}_2^\times$ be an element of infinite order which reduces to -1 modulo 4 – for example, $x = 3$ – and let K be the subgroup generated by x. The proof then proceeds in the same fashion.

Note that the arguments for parts (1) and (2) apply not only to $\mathbb{G}_n$, but also for any closed subgroup $G \subseteq \mathbb{G}_n$ which contains the center. In fact, for part (1) we need only have $C = \mathbb{F}_p^\times \subseteq G$.

3.) **There is a uniform and horizontal vanishing line at E_∞:** there is an integer N, depending only on n and p, so that in the Adams-Novikov Spectral Sequence (1.1.2) for any spectrum X

$$E_\infty^{s,*} = 0, \qquad s > N.$$

This can be found in the literature in several guises; for example, it can be put together from the material in Section 5 of [9], especially Lemma 5.11. See [3] §2.3 for references and explanation. See also [2] for even further explanation. If $p - 1 > n$, there is often a horizontal vanishing line at E_2. See Proposition 1.6 below.

1.1.2 Some local homological algebra.

Because E_0 is a complete local ring with maximal ideal $\mathfrak{m}$ generated by a regular sequence, we have a variety of tools from homological algebra. The classic paper here is Greenlees and May [15], but see also [24], Appendix A for direct connections to $E_*(-)$. Tensor product below will mean the $\mathfrak{m}$-completed tensor product. This is one place where the notation E_0 gets out of hand; thus we write $R = E_0$ in this subsection.

Let $u_0 = p$ and define a cochain complex $\Gamma_\mathfrak{m}$ by

$$\Gamma_\mathfrak{m} = \left(R \to R[\frac{1}{u_0}]\right) \otimes_R \left(R \to R[\frac{1}{u_1}]\right) \otimes_R \cdots \otimes_R \left(R \to R[\frac{1}{u_{n-1}}]\right)$$

and more generally we set

$$\Gamma_m(M) = M \otimes_R \Gamma_\mathfrak{m}.$$

Then $H^0_\mathfrak{m}(M) \stackrel{\text{def}}{=} H^0\Gamma_m(M)$ is the sub-module of $\mathfrak{m}$-torsion and we see that

$$H^s\Gamma_\mathfrak{m}(M) \stackrel{\text{def}}{=} H^s_\mathfrak{m}(M)$$

is the sth right derived functor of the $\mathfrak{m}$-torsion functor and thus independent of the choices. These are the local cohomology groups. If M is $\mathfrak{m}$-torsion, there is a composite functor spectral sequence

$$\mathrm{Ext}^p_R(M, H^q_{\mathfrak{m}}(N)) \Longrightarrow \mathrm{Ext}^{p+q}_R(M, N). \tag{1.1.5}$$

In the case $N = R$, this spectral sequence simplifies considerably. Note that $H^s_{\mathfrak{m}}(R) = 0$ unless $s = n$ and

$$H^n_{\mathfrak{m}}(R) \stackrel{\text{def}}{=} R/\mathfrak{m}^\infty \stackrel{\text{def}}{=} R/(p^\infty, u_1^\infty, \ldots, u_{n-1}^\infty)\ . \tag{1.1.6}$$

The R-module $R/\mathfrak{m}^\infty$ is an injective R-module and, in fact the injective hull of $R/\mathfrak{m}$. This is a consequence of Matlis duality for $(R, \mathfrak{m})$; see §12.1 of [7], especially Definition 12.1.2 and Remark 12.1.3.

Combining this observation with the spectral sequence (1.1.5) we have

$$\mathrm{Ext}^{p+n}_R(M, R) \cong \mathrm{Ext}^p_R(M, R/\mathfrak{m}^\infty) \cong \begin{cases} \mathrm{Hom}_R(M, R/\mathfrak{m}^\infty), & p = 0; \\ 0, & p \neq 0. \end{cases} \tag{1.1.7}$$

The module $R/\mathfrak{m}^\infty$ also arises in the theory of derived functors of completion. The completion functor

$$M \longmapsto \lim_k \ \left[M \otimes_R R/\mathfrak{m}^k\right]$$

is neither left nor right exact; however, it still has left derived functors $L^{\mathfrak{m}}_s(M)$. These vanish if $s > n$ and there is an isomorphism

$$\begin{aligned} L^{\mathfrak{m}}_n(M) &\cong \lim\ \mathrm{Tor}^R_n(M, R/\mathfrak{m}^k) \\ &\cong \lim\ \mathrm{Hom}_R(R/\mathfrak{m}^k, M) \\ &\cong \mathrm{Hom}_R(R/\mathfrak{m}^\infty, M). \end{aligned}$$

From this it follows that

$$L^{\mathfrak{m}}_s(M) \cong \mathrm{Ext}^{n-s}_R(R/\mathfrak{m}^\infty, M).$$

Remark 1.5 (**Morava modules**) If X is a spectrum we defined

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

By [24], Proposition 8.4, the E_*-module E_*X is $L^{\mathfrak{m}}$-complete; that is, the map

$$E_*X \longrightarrow L^{\mathfrak{m}}_0(E_*X)$$

is an isomorphism. In particular, E_*X is equipped with the $\mathfrak{m}$-adic topology.