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COMPLEX ALGEBRAIC THREEFOLDS

The first book on the explicit birational geometry of complex algebraic threefolds arising from the minimal model program, this text is sure to become an essential reference in the field of birational geometry. Threefolds remain the interface between low- and high-dimensional settings, and a good understanding of them is necessary in this actively evolving area.

Intended for advanced graduate students as well as researchers working in birational geometry, the book is as self-contained as possible. Detailed proofs are given throughout, and more than 100 examples help to deepen understanding of birational geometry.

The first part of the book deals with threefold singularities, divisorial contractions and flips. After a thorough explanation of the Sarkisov program, the second part is devoted to the analysis of outputs, specifically minimal models and Mori fibre spaces. The latter are divided into conical fibrations, del Pezzo fibrations and Fano threefolds according to the relative dimension.

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Complex Algebraic Threefolds

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For my family

Contents

Preface

The present book treats explicit aspects of the birational geometry of complex algebraic threefolds. It is a fundamental problem in birational geometry to find and analyse a good representative of each birational class of algebraic varieties. The minimal model program, or the MMP for short, conjecturally realises this by comparing the canonical divisors of varieties. According as the MMP has developed, the book has arisen from two perspectives.

Firstly, since the MMP in dimension three was established about a quarter century ago, it is desirable to understand individual threefolds explicitly by means of the MMP. The initial step is to describe birational transformations in the MMP. Now we have a practical classification of them in dimension three. The ensuing important subject is to analyse the threefolds output by the MMP. In this direction one can mention the Sarkisov program, which decomposes every birational map of Mori fibre spaces.

Secondly, the MMP in higher dimensions is still evolving actively after the existence of flips was proved, as typified in the settlement of the Borisov–Alexeev–Borisov conjecture. A good knowledge of threefolds is useful and will be necessary in further development of higher dimensional birational geometry. This is comparable to the nature that most results on threefolds are based upon the classical theory of surfaces.

The book concentrates on the explicit study of algebraic threefolds by the MMP. The author has tried to elucidate the proofs rigorously and to make the book as self-contained as possible. A number of examples will help to deepen the understanding of the reader. The reader is strongly encouraged to verify the computations in examples. Though it does not cover important topics such as affine geometry, derived categories and positive characteristic aspects, the book will supply enough knowledge of threefold birational geometry to enter the field of higher dimensional birational geometry.

The book is intended for advanced graduate students who are interested in the birational geometry of algebraic varieties. It can also be used by researchers as a reference for the classification results on threefolds. The reader should be familiar with basic algebraic geometry at the level of Hartshorne's textbook [178]. Some knowledge of the MMP is helpful but it is not a prerequisite. One can learn the general theory of the MMP from a standard book such as that by Kollár and Mori [277] or by Matsuki [307]. Whilst it roughly corresponds to Chapter 1 of the book, the main body starts from Chapter 2 and concentrates on threefolds.

The volume edited by Corti and Reid [97] is an outstanding collection from the same standpoint. It played a guiding role in the explicit study of algebraic threefolds when it was published. However, great progress has been made since then. The present book aims at an organised treatment of threefolds, including recent results. It seeks to be somewhat of a threefold version of the book on surfaces by Barth, Hulek, Peters and Van de Ven [30] or by Beauville [35].

Chapter 1 summarises the theory of the MMP in an arbitrary dimension but excludes detailed proofs. The first part, Chapters 2 to 5, of the main body deals with objects which appear in the course of the MMP. Chapter 2 classifies threefold singularities in the MMP completely. Then Chapters 3, 4 and 5 describe threefold birational transformations in the MMP, that is, divisorial contractions and flips. They contain all the necessary arguments by omitting only the parts which repeat the preceding arguments.

The second part, Chapters 6 to 10, is devoted to the analysis of outputs of the MMP, which are Mori fibre spaces and minimal models. After Chapter 6 explains the general theory of the Sarkisov program, Chapters 7, 8 and 9 investigate the geometry of threefold Mori fibre spaces according to the relative dimension of the fibre structure. Finally Chapter 10 discusses minimal threefolds from the point of view of abundance.

The author would like to express thanks to all the colleagues with whom he held discussions. Yujiro Kawamata introduced him to the subject of birational geometry as his academic supervisor. Alessio Corti and Miles Reid communicated their profound knowledge of threefolds to him during his visit to the University of Cambridge. When he stayed at the Institute for Advanced Study, he received warm hospitality from János Kollár. He also learnt a great deal from the regular seminar organised by Shigefumi Mori, Shigeru Mukai and Noboru Nakayama. Finally he would like to thank Philip Meyler and John Linglei Meng at Cambridge University Press for their support for the publication.