1

The Minimal Model Program

This chapter outlines the general theory of the minimal model program. We shall study algebraic threefolds thoroughly in the subsequent chapters in alignment with the program. The reader who is not familiar with the program may grasp the basic notions at first and refer back later.

Blowing up a surface at a point is not an essential operation from the birational point of view. Its exceptional curve is characterised numerically as a (-1)-curve. As is the case in this observation, the intersection number is a basic linear tool in birational geometry. The minimal model program, or the MMP for short, outputs a representative of each birational class that is minimal with respect to the numerical class of the canonical divisor.

The MMP grew out of the surface theory with allowing mild singularities. For a given variety, it produces a minimal model or a Mori fibre space after finitely many birational transformations, which are divisorial contractions and flips. Now the program is formulated in the logarithmic framework where we treat a pair consisting of a variety and a divisor.

The MMP functions subject to the existence and termination of flips. Hacon and McKernan with Birkar and Cascini proved the existence of flips in an arbitrary dimension. Considering a flip to be the relative canonical model, they established the MMP with scaling in the birational setting. The termination of threefold flips follows from the decrease in the number of divisors with small log discrepancy. Shokurov reduced the termination in an arbitrary dimension to certain conjectural properties of the minimal log discrepancy.

It is also important to analyse the representative output by the MMP. The Sarkisov program decomposes a birational map of Mori fibre spaces into elementary ones. For a minimal model, we expect the abundance which claims the freedom of the linear system of a multiple of the canonical divisor. It defines a morphism to the projective variety associated with the canonical ring, which we know is finitely generated.

1.1 Preliminaries

We shall fix the notation and recall the fundamentals of algebraic geometry. The book [178] by Hartshorne is a standard reference.

The *natural numbers* begin with zero. The symbol $R_{\geq r}$ for $R = \mathbf{N}, \mathbf{Z}, \mathbf{Q}$ or $\mathbf{R}$ stands for the subset $\{x \in R \mid x \geq r\}$ and similarly $R_{>r} = \{x \in R \mid x > r\}$. For instance, $\mathbf{N} = \mathbf{Z}_{\geq 0}$. The quotient $\mathbf{Z}_r = \mathbf{Z}/r\mathbf{Z}$ is the *cyclic group* of order r. The *round-down* $\lfloor r \rfloor$ of a real number r is the greatest integer less than or equal to r, whilst the *round-up* $\lceil r \rceil$ is defined as $\lceil r \rceil = -\lfloor -r \rfloor$.

Schemes A *scheme* is always assumed to be separated. It is said to be *integral* if it is irreducible and reduced.

We work over the field $\mathbf{C}$ of complex numbers unless otherwise mentioned. An *algebraic scheme* is a scheme of finite type over $\operatorname{Spec} k$ for the algebraically closed ground field k, which is tacitly assumed to be $\mathbf{C}$. We call it a *complex scheme* when we emphasise that it is defined over $\mathbf{C}$. An algebraic scheme is said to be *complete* if it is proper over $\operatorname{Spec} k$. A *point* in an algebraic scheme usually means a closed point.

A *variety* is an integral algebraic scheme. A *complex variety* is a variety over $\mathbf{C}$. A *curve* is a variety of dimension one and a *surface* is a variety of dimension two. An *n-fold* is a variety of dimension n. The *affine space* $\mathbf{A}^n$ is $\operatorname{Spec} k[x_1, \dots, x_n]$ and the *projective space* $\mathbf{P}^n$ is $\operatorname{Proj} k[x_0, \dots, x_n]$. The origin of $\mathbf{A}^n$ is denoted by o.

The *germ* $x \in X$ of a scheme is considered at a closed point unless otherwise specified. It is an equivalence class of the pair (X, x) of a scheme X and a point x in X where (X, x) is equivalent to (X', x') if there exists an isomorphism $U \simeq U'$ of open neighbourhoods $x \in U \subset X$ and $x' \in U' \subset X'$ sending x to x'. By a *singularity*, we mean the germ at a singular point as a rule.

For a locally free coherent sheaf $\mathscr{E}$ on an algebraic scheme X, the *projective space bundle* $\mathbf{P}(\mathscr{E}) = \operatorname{Proj}_X S\mathscr{E}$ over X is defined by the symmetric $\mathscr{O}_X$-algebra $S\mathscr{E} = \bigoplus_{i \in \mathbf{N}} S^i\mathscr{E}$ of $\mathscr{E}$. It is a $\mathbf{P}^n$*-bundle* if $\mathscr{E}$ is of rank $n+1$. In particular, the projective space $\mathbf{P}V = \operatorname{Proj} SV$ is defined for a finite dimensional vector space V. It is regarded as the quotient space $(V^\vee \setminus 0)/k^\times$ of the dual vector space $V^\vee$ minus zero by the action of the multiplicative group $k^\times = k \setminus \{0\}$ of the ground field k. As used above, the symbol $^\vee$ stands for the dual and $^\times$ for the group of units.

Morphisms For a morphism $\pi \colon X \to Y$ of schemes, the *image* $\pi(A)$ of a closed subset A of X and the *inverse image* $\pi^{-1}(B)$ of a closed subset B of Y are considered set-theoretically. When π is proper and A is a closed subscheme, we regard $\pi(A)$ as a reduced scheme. We also regard $\pi^{-1}(B)$ for a closed subscheme B as a reduced scheme and distinguish it from the scheme-theoretic fibre $X \times_Y B$.

A *rational map* $f\colon X \dashrightarrow Y$ of algebraic schemes is an equivalence class of a morphism $U \to Y$ defined on a dense open subset U of X. The *image* $f(X)$ of f is the image $p(\Gamma)$ of the graph Γ of f as a closed subscheme of $X \times Y$ by the projection $p\colon X \times Y \to Y$. We say that a morphism or a rational map is *birational* if it has an inverse as a rational map. Two algebraic schemes are *birational* if there exists a birational map between them. By definition, two varieties are birational if and only if they have the same function field.

Let $\pi\colon X \to Y$ be a morphism of algebraic schemes. We say that π is *projective* if it is isomorphic to $\operatorname{Proj}_Y \mathscr{R} \to Y$ by a graded $\mathscr{O}_Y$-algebra $\mathscr{R} = \bigoplus_{i \in \mathbf{N}} \mathscr{R}_i$ generated by coherent $\mathscr{R}_1$, with $\mathscr{R}_0 = \mathscr{O}_Y$. When Y is quasi-projective, the projectivity of π means that it is realised as a closed subscheme of a relative projective space $\mathbf{P}^n \times Y \to Y$. An invertible sheaf $\mathscr{L}$ on X is *relatively very ample* (or *very ample* over Y or π*-very ample*) if it is isomorphic to $\mathscr{O}(1)$ by an expression $X \simeq \operatorname{Proj}_Y \mathscr{R}$ as above. We say that $\mathscr{L}$ is *relatively ample* (π*-ample*) if $\mathscr{L}^{\otimes a}$ is relatively very ample for some positive integer a.

Suppose that $\pi\colon X \to Y$ is proper. We say that π *has connected fibres* if the natural map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is an isomorphism. This implies that the fibre $X \times_Y y$ at every $y \in Y$ is connected and non-empty [160, III corollaire 4.3.2]. The proof for a projective morphism is in [178, III corollary 11.3]. In general, π admits the *Stein factorisation* $\pi = g \circ f$ with $f\colon X \to Z$ and $g\colon Z \to Y$ defined by $Z = \operatorname{Spec}_Y \pi_*\mathscr{O}_X$, for which f is proper with connected fibres and g is finite. If π is a proper birational morphism from a variety to a normal variety, then the factor g in the Stein factorisation is an isomorphism and hence π has connected fibres. This is referred to as *Zariski's main theorem*.

Lemma 1.1.1 *Let $\pi\colon X \to Y$ and $\varphi\colon X \to Z$ be morphisms of algebraic schemes such that π is proper and has connected fibres. If every curve in X contracted to a point by π is also contracted by φ, then φ factors through π as $\varphi = f \circ \pi$ for a morphism $f\colon Y \to Z$.*

Proof Let Y^m and Z^m denote the sets of closed points in Y and Z respectively. For $y \in Y^m$, the inverse image $\pi^{-1}(y)$ is connected and $\varphi(\pi^{-1}(y))$ is one point. Define $f^m\colon Y^m \to Z^m$ by $f^m(y) = \varphi(\pi^{-1}(y))$. Since π is proper and surjective, for any closed subset B of Z, $\pi(\varphi^{-1}(B))$ is closed in Y and $(f^m)^{-1}(B|_{Z^m}) = \pi(\varphi^{-1}(B))|_{Y^m}$. Thus f^m extends to a continuous map $f\colon Y \to Z$, which is a morphism of schemes by the natural map $\mathscr{O}_Z \to \varphi_*\mathscr{O}_X = f_*\pi_*\mathscr{O}_X = f_*\mathscr{O}_Y$. □

Chow's lemma [160, II §5.6] replaces the proper morphism $\pi\colon X \to Y$ by a projective morphism. It asserts the existence of a projective birational morphism $\mu\colon X' \to X$ such that $\pi \circ \mu\colon X' \to Y$ is projective. The *projection formula* and the *Leray spectral sequence*, formulated for ringed spaces in [160, 0 §12.2], will be frequently used. The reference [198, section 3.6] explains spectral sequences from our perspective.

Theorem 1.1.2 (Projection formula) *Let $\pi\colon X \to Y$ be a morphism of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and let $\mathscr{E}$ be a finite locally free $\mathscr{O}_Y$-module. Then there exists a natural isomorphism $R^i\pi_*\mathscr{F} \otimes \mathscr{E} \simeq R^i\pi_*(\mathscr{F} \otimes \pi^*\mathscr{E})$.*

Theorem 1.1.3 (Leray spectral sequence) *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then there exists a spectral sequence*

$$E_2^{p,q} = R^p g_* R^q f_* \mathscr{F} \Rightarrow E^{p+q} = R^{p+q}(g \circ f)_* \mathscr{F}.$$

In practice for a spectral sequence $E_2^{p,q} \Rightarrow E^{p+q}$, we assume that $E_2^{p,q}$ is zero whenever p or q is negative. Then there exists an exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2.$$

If further $E_2^{p,q} = 0$ for all $p \geq 0$ and $q \geq 1$, then $E_2^{p,0} \simeq E^p$. Likewise if $E_2^{p,q} = 0$ for all $p \geq 1$ and $q \geq 0$, then $E_2^{0,q} \simeq E^q$.

Cohomologies We write $H^i(\mathscr{F})$ for the cohomology $H^i(X, \mathscr{F})$ of a sheaf $\mathscr{F}$ of abelian groups on a topological space X when there is no confusion. If X is noetherian, then $H^i(\mathscr{F})$ vanishes for all i greater than the dimension of X.

Let $\mathscr{F}$ be a coherent sheaf on an algebraic scheme X. If X is affine, then $H^i(\mathscr{F}) = 0$ for all $i \geq 1$. If $\pi\colon X \to Y$ is a proper morphism, then the higher direct image $R^i\pi_*\mathscr{F}$ is coherent [160, III théorème 3.2.1]. In particular if X is complete, then $H^i(\mathscr{F})$ is a finite dimensional vector space. The dimension of $H^i(\mathscr{F})$ is denoted by $h^i(\mathscr{F})$. The alternating sum $\chi(\mathscr{F}) = \sum_{i \in \mathbf{N}} (-1)^i h^i(\mathscr{F})$ is called the *Euler characteristic* of $\mathscr{F}$.

Let X be a complete scheme of dimension n. For a coherent sheaf $\mathscr{F}$ and an invertible sheaf $\mathscr{L}$ on X, the *asymptotic Riemann–Roch theorem* defines the *intersection number* $(\mathscr{L}^n \cdot \mathscr{F}) \in \mathbf{Z}$ by the expression

$$\chi(\mathscr{L}^{\otimes l} \otimes \mathscr{F}) = \frac{(\mathscr{L}^n \cdot \mathscr{F})}{n!} l^n + O(l^{n-1}),$$

where by *Landau's symbol* O, $f(l) = O(g(l))$ means the existence of a constant c such that $|f(l)| \leq c|g(l)|$ for any large l. By this, Grothendieck's *dévissage* yields the estimate $h^i(\mathscr{F} \otimes \mathscr{L}^{\otimes l}) = O(l^n)$ for all i [266, section VI.2].

If X is projective with a very ample sheaf $\mathscr{O}_X(1)$, then the Euler characteristic $\chi(\mathscr{F} \otimes \mathscr{O}_X(l))$ is described as a polynomial in $\mathbf{Q}[l]$, called the *Hilbert* *polynomial* of $\mathscr{F}$. The vanishing of $H^i(\mathscr{F} \otimes \mathscr{O}_X(l))$ below is known as *Serre vanishing*.

Theorem 1.1.4 (Serre) *Let $\mathscr{F}$ be a coherent sheaf on a projective scheme X. Then for any sufficiently large integer l, the twisted sheaf $\mathscr{F} \otimes \mathscr{O}_X(l)$ is generated by global sections and satisfies $H^i(\mathscr{F} \otimes \mathscr{O}_X(l)) = 0$ for all $i \geq 1$.*

We have the *cohomology and base change theorem* for flat families of coherent sheaves [160, III §§7.6–7.9], [361, section 5]. See also [178, section III.12].

Theorem 1.1.5 (Cohomology and base change) *Let $\pi\colon X \to T$ be a proper morphism of algebraic schemes. Let $\mathscr{F}$ be a coherent sheaf on X flat over T. Take the restriction $\mathscr{F}_t$ of $\mathscr{F}$ to the fibre $X_t = X \times_T t$ at a closed point t in T and consider the natural map*

$$\alpha_t^i\colon R^i\pi_*\mathscr{F} \otimes k(t) \to H^i(X_t, \mathscr{F}_t),$$

where $k(t)$ is the skyscraper sheaf of the residue field at t.

(i) *The dimension $h^i(\mathscr{F}_t)$ is upper semi-continuous on T and the Euler characteristic $\chi(\mathscr{F}_t)$ is locally constant on T.*
(ii) *Fix i and t and suppose that α_t^i is surjective. Then $\alpha_{t'}^i$ is an isomorphism for all t' in a neighbourhood at t. Further, $R^i\pi_*\mathscr{F}$ is locally free at t if and only if α_t^{i-1} is surjective.*
(iii) (Grauert) *Suppose that T is reduced. Fix i. If $h^i(\mathscr{F}_t)$ is locally constant, then $R^i\pi_*\mathscr{F}$ is locally free and α_t^i is an isomorphism.*

Divisors Let X be an algebraic scheme. We write $\mathscr{K}_X$ for the sheaf of total quotient rings of $\mathscr{O}_X$. If X is a variety, then it is the constant sheaf of the function field $K(X)$ of X. A *Cartier divisor* D on X is a global section of the quotient sheaf $\mathscr{K}_X^\times/\mathscr{O}_X^\times$ of multiplicative groups of units. It is associated with an invertible subsheaf $\mathscr{O}_X(D)$ of $\mathscr{K}_X$. If D is represented by local sections $f_i \in \mathscr{K}_{U_i}^\times$ with $f_i f_j^{-1} \in \mathscr{O}_{U_i \cap U_j}^\times$, then $\mathscr{O}_X(D)|_{U_i} = f_i^{-1}\mathscr{O}_{U_i}$. We say that D is *principal* if it is defined by a global section of $\mathscr{K}_X^\times$ or equivalently $\mathscr{O}_X(D) \simeq \mathscr{O}_X$. The principal divisor given by $f \in \Gamma(X, \mathscr{K}_X^\times)$ is denoted by $(f)_X$. If f_i belongs to $\mathscr{O}_{U_i} \cap \mathscr{K}_{U_i}^\times$ for all i, then D defines a closed subscheme of X and we say that D is *effective*.

The *Picard group* Pic X of X is the group of isomorphism classes of invertible sheaves on X. It has an isomorphism

$$\operatorname{Pic} X \simeq H^1(\mathscr{O}_X^\times).$$

In fact this holds for any ringed space via Čech cohomology. The proof is found in [440, section 5.4]. The isomorphism for a variety X is derived at once from the vanishing of $H^1(\mathscr{K}_X^\times)$ for the flasque sheaf $\mathscr{K}_X^\times$.

By *Serre's criterion*, an algebraic scheme X is normal if and only if it satisfies the conditions R_1 and S_2 defined as

(R_i) for any $\eta \in X$, $\mathscr{O}_{X,\eta}$ is regular if $\mathscr{O}_{X,\eta}$ is of dimension at most i and
(S_i) for any $\eta \in X$, $\mathscr{O}_{X,\eta}$ is Cohen–Macaulay if $\mathscr{O}_{X,\eta}$ is of depth less than i,

in which we consider scheme-theoretic points $\eta \in X$. Let X be a normal variety. A closed subvariety of codimension one in X is called a *prime divisor*. A *Weil divisor* D on X, or simply called a *divisor*, is an element in the free abelian group $Z^1(X)$ generated by prime divisors on X. A Cartier divisor on a normal variety is a Weil divisor. Every Weil divisor on a smooth variety is Cartier. The divisor D is expressed as a finite sum $D = \sum_i d_i D_i$ of prime divisors D_i with non-zero integers d_i. The *support* of D is the union of D_i. The divisor D is *effective* if all d_i are positive, and it is *reduced* if all d_i equal one. We write $D \le D'$ if $D' - D$ is effective. The *linear equivalence* $D \sim D'$ of divisors means that $D' - D$ is principal.

The divisor D is associated with a divisorial sheaf $\mathscr{O}_X(D)$ on X. A *divisorial* sheaf is a reflexive sheaf of rank one, where a coherent sheaf $\mathscr{F}$ is said to be *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ to the double dual is an isomorphism. The sheaf $\mathscr{O}_X(D)$ is the subsheaf of $\mathscr{K}_X$ defined by

$$\Gamma(U, \mathscr{O}_X(D)) = \{f \in K(X) \mid (f)_U + D|_U \ge 0\},$$

in which zero is contained in the set on the right by convention. The *divisor class group* $\operatorname{Cl} X$ is the quotient of the group $Z^1(X)$ of Weil divisors divided by the subgroup of principal divisors. It is regarded as the group of isomorphism classes of divisorial sheaves on X and has an injection $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$.

Linear systems Let X be a normal complete variety. Let D be a Weil divisor on X and let V be a vector subspace of global sections in $H^0(\mathscr{O}_X(D))$. The projective space $\Lambda = \mathbf{P}V^\vee = (V \setminus 0)/k^\times$ where k is the ground field is called a *linear system* on X. It defines a rational map $X \dashrightarrow \mathbf{P}V$. When $V = H^0(\mathscr{O}_X(D))$, we write $|D| = \mathbf{P}H^0(\mathscr{O}_X(D))^\vee$ and call it a *complete* linear system. By the inclusion $\mathscr{O}_X(D) \subset \mathscr{K}_X$, the linear system $|D|$ is regarded as the set of effective divisors D' linearly equivalent to D, and Λ is a subset of $|D|$. That is,

$$\Lambda \subset |D| = \{D' \ge 0 \mid D' \sim D\}.$$

The *base locus* of Λ means the scheme-theoretic intersection $B = \bigcap_{D' \in \Lambda} D'$ in X. We say that the linear system Λ is *free* if B is empty. We say that Λ is *mobile* if B is of codimension at least two. The divisor D is said to be *free* (resp.

mobile) if $|D|$ is free (resp. mobile). By definition, a free Weil divisor is Cartier. When $\emptyset \neq \Lambda \subset |D|$, Λ is decomposed as $\Lambda = \Lambda' + F$ with a mobile linear system $\Lambda' \subset |D - F|$ and the maximal effective divisor F such that $F \leq D_1$ for all $D_1 \in \Lambda$. The constituents Λ' and F are called the *mobile part* and the *fixed part* of Λ respectively. The rational map defined by Λ' is isomorphic to $X \dashrightarrow \mathbf{P}V$. The linear system Λ is mobile if and only if F is zero.

Even if X is not complete, the linear system $\Lambda = \mathbf{P}V^\vee$ is defined for a finite dimensional vector subspace V of $H^0(\mathscr{O}_X(D))$. We consider $|D|$ to be the direct limit $\varinjlim_V \Lambda$ of linear systems.

A *general* point in a variety Z means a point in a dense open subset U of Z. A *very general* point in Z means a point in the intersection $\bigcap_{i \in \mathbf{N}} U_i$ of countably many dense open subsets U_i. Thus by the general member of the linear system Λ, we mean a general point in Λ as a projective space. Bertini's theorem asserts that a free linear system on a smooth complex variety has a smooth member. The statement for the hyperplane section holds even in positive characteristic.

Theorem 1.1.6 (Bertini's theorem) *Let $\Lambda = \mathbf{P}V^\vee$ be a free linear system on a smooth variety X and let $\varphi\colon X \to \mathbf{P}V$ be the induced morphism. Suppose that φ is a closed embedding or the ground field is of characteristic zero. Then the general member H of Λ is a smooth divisor on X, and if the image $\varphi(X)$ is of dimension at least two, then H is a smooth prime divisor.*

The canonical divisor It is the canonical divisor that plays the most important role in birational geometry. The *sheaf of differentials* on an algebraic scheme X is denoted by Ω_X. When X is smooth, Ω_X^i denotes the i-th exterior power $\bigwedge^i \Omega_X$.

Definition 1.1.7 The *canonical divisor* K_X on a normal variety X is the divisor defined up to linear equivalence by the isomorphism $\mathscr{O}_X(K_X)|_U \simeq \Omega_U^n$ on the smooth locus U in X, where n is the dimension of X.

Example 1.1.8 The projective space $\mathbf{P}^n$ has the canonical divisor $K_{\mathbf{P}^n} \sim -(n+1)H$ for a hyperplane H. This follows from the *Euler sequence*

$$0 \to \Omega_{\mathbf{P}^n} \to \mathscr{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbf{P}^n} \to 0.$$

One can describe $K_{\mathbf{P}^n}$ in an explicit way. Take homogeneous coordinates $x_0, \dots, x_n$ of $\mathbf{P}^n$. Let $U_i \simeq \mathbf{A}^n$ denote the complement of the hyperplane H_i defined by x_i. The chart U_0 admits a nowhere vanishing n-form $dy_1 \wedge \cdots \wedge dy_n$ with coordinates $y_1, \dots, y_n$ for $y_i = x_i x_0^{-1}$. It is expressed on the chart U_1 having coordinates $z_0, z_2, \dots, z_n$ for $z_i = x_i x_1^{-1}$ as the rational n-form $dz_0^{-1} \wedge d(z_2 z_0^{-1}) \wedge \cdots \wedge d(z_n z_0^{-1}) = -z_0^{-(n+1)} dz_0 \wedge dz_2 \wedge \cdots \wedge dz_n$, which has pole of order $n+1$ along H_0. Thus $K_{\mathbf{P}^n} \sim -(n+1)H_0$.

In spite of the ambiguity concerning linear equivalence, it is standard to treat the canonical divisor as if it were a specified divisor.

For a closed subscheme D of an algebraic scheme X, there exists an exact sequence $\mathscr{I}/\mathscr{I}^2 \to \Omega_X \otimes \mathscr{O}_D \to \Omega_D \to 0$, where $\mathscr{I}$ is the ideal sheaf in $\mathscr{O}_X$ defining D. This induces the *adjunction formula*, which connects the canonical divisor to that on a Cartier divisor.

Theorem 1.1.9 (Adjunction formula) *Let X be a normal variety and let D be a reduced Cartier divisor on X which is normal. Then $K_D = (K_X + D)|_D$ in the sense that $\mathscr{O}_D(K_D) \simeq \mathscr{O}_X(K_X + D) \otimes \mathscr{O}_D$.*

Duality Albeit Grothendieck's duality theory works in the derived category for proper morphisms [177], it is extremely hard to obtain the dualising complex and a trace map in a compatible manner. The theory becomes efficient if it is restricted to the Cohen–Macaulay projective case as explained in [178, section III.7] and [277, section 5.5]. For example, the dualising complex on a Cohen–Macaulay projective scheme X of pure dimension n is the shift $\omega_X[n]$ of the dualising sheaf ω_X.

Definition 1.1.10 Let X be a complete scheme of dimension n over an algebraically closed field k. The *dualising sheaf* ω_X for X is a coherent sheaf on X endowed with a *trace map* $t\colon H^n(\omega_X) \to k$ such that for any coherent sheaf $\mathscr{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathscr{F}, \omega_X) \times H^n(\mathscr{F}) \to H^n(\omega_X) \xrightarrow{t} k$$

induces an isomorphism $\mathrm{Hom}(\mathscr{F}, \omega_X) \simeq H^n(\mathscr{F})^\vee$.

The dualising sheaf is unique up to isomorphism if it exists. The projective space $\mathbf{P}^n$ has the dualising sheaf $\omega_{\mathbf{P}^n} \simeq \bigwedge^n \Omega_{\mathbf{P}^n}$. This with Lemma 1.1.11 yields the existence of ω_X for every projective scheme X by taking a finite morphism $X \to \mathbf{P}^n$ known as projective *Noether normalisation*. If X is embedded into a projective space P with codimension r, then $\omega_X \simeq \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$ [178, III proposition 7.5]. If X is a normal projective variety, then ω_X coincides with the sheaf $\mathscr{O}_X(K_X)$ associated with the canonical divisor.

For a finite morphism $\pi\colon X \to Y$ of algebraic schemes, the push-forward π_* defines an equivalence of categories from the category of coherent $\mathscr{O}_X$-modules to that of coherent $\pi_*\mathscr{O}_X$-modules. This associates every coherent sheaf $\mathscr{G}$ on Y functorially with a coherent sheaf $\pi^!\mathscr{G}$ on X satisfying $\pi_* \mathscr{H}om_X(\mathscr{F}, \pi^!\mathscr{G}) \simeq \mathscr{H}om_Y(\pi_*\mathscr{F}, \mathscr{G})$ for any coherent sheaf $\mathscr{F}$ on X.

Lemma 1.1.11 *Let $\pi\colon X \to Y$ be a finite morphism of complete schemes of the same dimension. If the dualising sheaf ω_Y for Y exists, then $\omega_X = \pi^!\omega_Y$ is the dualising sheaf for X.*

Proof Let n denote the common dimension of X and Y. For a coherent sheaf $\mathscr{F}$ on X, $\operatorname{Hom}_X(\mathscr{F}, \pi^!\omega_Y) = \operatorname{Hom}_Y(\pi_*\mathscr{F}, \omega_Y)$ is dual to $H^n(\mathscr{F}) = H^n(\pi_*\mathscr{F})$ by the property of ω_Y, where the latter equality follows from the Leray spectral sequence $H^p(R^q\pi_*\mathscr{F}) \Rightarrow H^{p+q}(\mathscr{F})$. □

The duality for Cohen–Macaulay sheaves on a projective scheme is derived from that on the projective space via projective Noether normalisation. See [277, theorem 5.71].

Theorem 1.1.12 (Serre duality) *Let X be a projective scheme of dimension n. Let $\mathscr{F}$ be a Cohen–Macaulay coherent sheaf on X with support of pure dimension n. Then $H^i(\mathscr{H}om_X(\mathscr{F}, \omega_X))$ is dual to $H^{n-i}(\mathscr{F})$ for all i.*

The *adjunction formula* $\omega_D \simeq \omega_X \otimes \mathscr{O}_X(D) \otimes \mathscr{O}_D$ holds for a Cohen–Macaulay projective scheme X of pure dimension and an effective Cartier divisor D on X. Compare it with Theorem 1.1.9.

Resolution of singularities A projective birational morphism is described as a blow-up. The *blow-up* of an algebraic scheme X along a coherent ideal sheaf $\mathscr{I}$ in $\mathscr{O}_X$, or along the closed subscheme defined by $\mathscr{I}$, is the projective morphism $\pi\colon B = \operatorname{Proj}_X \bigoplus_{i\in\mathbf{N}} \mathscr{I}^i \to X$. The pull-back $\mathscr{I}\mathscr{O}_B = \pi^{-1}\mathscr{I} \cdot \mathscr{O}_B$ in $\mathscr{O}_B$ is an invertible ideal sheaf. Notice that $\mathscr{I}\mathscr{O}_B$ is different from $\pi^*\mathscr{I}$. The blow-up π has the universal property that every morphism $\varphi\colon Y \to X$ that makes $\mathscr{I}\mathscr{O}_Y$ invertible factors through π as $\varphi = \pi \circ f$ for a morphism $f\colon Y \to B$.

Let $f\colon X \dashrightarrow Y$ be a birational map of varieties. The *exceptional locus* of f is the locus in X where f is not biregular. Let Z be a closed subvariety of X not contained in the exceptional locus of f. The *strict transform* f_*Z in Y of Z is the closure of the image of $Z \dashrightarrow Y$. When X and Y are normal, the *strict transform* f_*P in Y of an arbitrary prime divisor P on X is defined as a divisor in such a manner that f_*P is zero if P is in the exceptional locus of f. By linear extension, we define the strict transform f_*D in Y for any divisor D on X.

Resolution of singularities is a fundamental tool in complex birational geometry. We say that a reduced divisor D on a smooth variety X is *simple normal crossing*, or *snc* for short, if D is defined at every point x in X by the product $x_1 \cdots x_m$ of a part of a regular system $x_1, \dots, x_n$ of parameters in $\mathscr{O}_{X,x}$.

Definition 1.1.13 A *resolution* of a variety X is a projective birational morphism $\mu\colon X' \to X$ from a smooth variety. The resolution μ is said to be *strong* if it is isomorphic on the smooth locus in X.

Definition 1.1.14 Let X be a normal variety, let Δ be a divisor on X and let $\mathscr{I}$ be a coherent ideal sheaf in $\mathscr{O}_X$. A *log resolution* of $(X, \Delta, \mathscr{I})$ is a resolution $\mu\colon X' \to X$ such that

- the exceptional locus E of μ is a divisor on X',
- the pull-back $\mathscr{I}\mathscr{O}_{X'}$ is invertible and hence defines a divisor D and
- $E + D + \mu_*^{-1}S$ has snc support for the support S of Δ.

The log resolution μ is said to be *strong* if it is isomorphic on the maximal locus U in X such that U is smooth, $\mathscr{I}|_U$ defines a divisor D_U and $D_U + S|_U$ has snc support. A (strong) log resolution of X means that of $(X, 0, \mathscr{O}_X)$, and those of (X, Δ) and $(X, \mathscr{I})$ are likewise defined.

The existence of these resolutions for complex varieties is due to Hironaka. The items (i) and (ii) below are derived from the main theorems I and II in [187] respectively.

Theorem 1.1.15 (Hironaka [187]) (i) *A strong resolution exists for every complex variety.*

(ii) *A strong log resolution exists for every pair $(X, \mathscr{I})$ of a smooth complex variety X and a coherent ideal sheaf $\mathscr{I}$ in $\mathscr{O}_X$.*

Hironaka's construction includes the existence of a strong log resolution $X' \to X$ equipped with an effective exceptional divisor E on X' such that $\mathscr{O}_{X'}(-E)$ is relatively ample.

Analytic spaces We shall occasionally consider a complex scheme to be an analytic space in the Euclidean topology. Whilst an algebraic scheme is obtained by gluing affine schemes in $\mathbf{A}^n$, an analytic space is constructed by gluing analytic models in a domain in $\mathbf{C}^n$. A reference is [151]. The ring of convergent complex power series is denoted by $\mathbf{C}\{x_1, \ldots, x_n\}$.

Let D be a domain in the complex manifold $\mathbf{C}^n$. Let $\mathscr{O}_D$ denote the sheaf of holomorphic functions on D. Let $\mathscr{I}$ be an ideal sheaf in $\mathscr{O}_D$ generated by a finite number of global sections. The locally $\mathbf{C}$-ringed space $(V, (\mathscr{O}_D/\mathscr{I})|_V)$ for the support V of the quotient sheaf $\mathscr{O}_D/\mathscr{I}$ is called an *analytic model*, where being $\mathbf{C}$-*ringed* means having the structure sheaf of $\mathbf{C}$-algebras. An *analytic space* is a locally $\mathbf{C}$-ringed Hausdorff space such that every point has an open neighbourhood isomorphic to an analytic model.

Every complex scheme X is associated with an analytic space X_h. This defines a functor h from the category of complex schemes to the category of analytic spaces. There exists a natural morphism $X_h \to X$ of locally $\mathbf{C}$-ringed spaces which maps X_h bijectively to the set of closed points in X. It pulls back a coherent sheaf $\mathscr{F}$ on X to a coherent sheaf $\mathscr{F}_h$ on X_h. When X is complete, it