1

FUNDAMENTAL CONCEPTS

1.1 Introduction

The object of this chapter is to lay out the principal ideas and nomenclature of group theory in preparation for the physical applications discussed in later chapters. We shall look at what group theory deals with, we shall define the mathematical meaning of a group, we shall show examples of several groups, and we shall discuss the key subject of matrix representations of groups (with an example). A review of matrix algebra and definitions of some special matrices concludes the chapter.

As a student of science, you spent several years studying calculus, differential equations, and the properties of important mathematical functions (trigonometric functions, exponentials, Bessel functions, etc.). You used these tools to solve problems in Newtonian mechanics, electromagnetism, and maybe even problems in quantum mechanics.

At its heart, group theory is very different from calculus. It is more abstract and more fundamental, with little reliance on explicit mathematical functions. As we shall see in this text, group theory, though abstract, nevertheless has great power in dealing with a wide range of physical phenomena. One example is the angular momentum ("spin") of an electron that is experimentally a dimensionless "point" particle with no analogue in Newtonian mechanics.

In physical applications, group theory calculates numerical results by using mathematical functions in the group's representation matrices.

More profoundly, group theory can give deeper insight into subjects you may have already studied; for instance, the conservation of energy and the structure of hydrogen-atom wave functions in quantum mechanics. Newton invented calculus to explain how forces acting on an object determine its motion. In modern high-energy particle physics the forces are not well known, yet group theory in its abstract generality provides predictive schemes for classifying "strange" particles.

P. W. Anderson (1923–2020, Nobel laureate in physics 1977) wrote "It is only slightly overstating the case to say that physics is the study of symmetry."

1.2 Operations

Group theory deals with *operations*, also called *transformations*. In this book the symbols for operations are written in bold. We use the convention that an operation operates on the object (the *operand*) to its right.

Consider the simple example of a transformation $\mathbf{T}$ that operates on a variable x (the operand) to change its sign to $-x$. This is written symbolically as

$$\mathbf{T}x = -x.$$

If $\mathbf{T}$ operates on the function $ax + b$, where a and b are constants, $\mathbf{T}$ operates only on x and has no effect on constants. Hence

$$\begin{aligned}\mathbf{T}(ax + b) &= a\mathbf{T}x + b \\ &= -ax + b.\end{aligned}$$

If $\mathbf{T}$ operates twice in succession, the sequence $\mathbf{TT}$ can be written as $\mathbf{T}^2$:

$$\begin{aligned}\mathbf{T}^2 x &= \mathbf{TT}x \\ &= \mathbf{T}(\mathbf{T}x) \\ &= \mathbf{T}(-x) \\ &= x.\end{aligned}$$

An even simpler operator is the *identity* operator, which produces no change in the operand. The identity operator in group theory is conventionally given the symbol $\mathbf{E}$, from the German *Einheit*, unity or, literally, oneness:

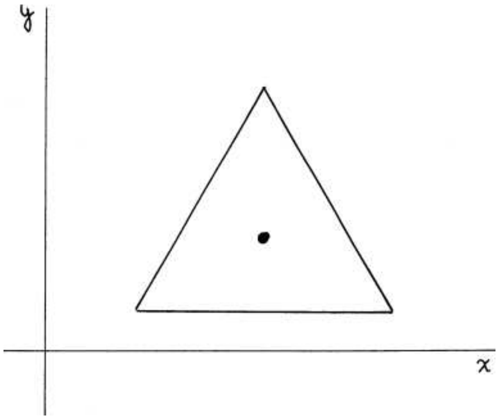
$$\mathbf{E}x = x.$$

These simple examples illustrate the abstract nature of group theory. The operators are not expressed in terms of explicit mathematical functions; instead, operators are defined in terms of their effect on the operand.

1.2.1 Symmetry Operations

The figure shows an equilateral triangle in the x-y plane. The dot marks the location of the triangle's geometric center, the point equally distant from all three apexes.

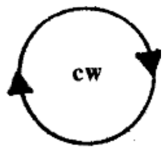
Consider now three operations, $\mathbf{E}$, $\mathbf{A}$, and $\mathbf{B}$, that rotate the triangle about its geometric center through the specified angles.

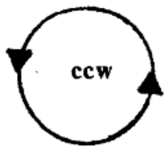


E: $rotate\,by\;0°\;(equivalently,\;rotate\,by\;360°)$
A: $rotate\,by\;120°$
B: $rotate\,by\;240°$

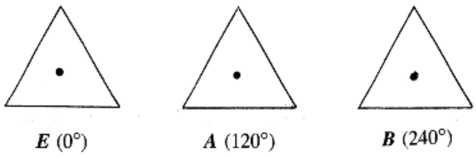
As the notation implies, operation **E** (rotation by 0°) clearly plays the role of the identity operation.

In this digital age, clocks with hands are no longer common but the terms *counterclockwise* and *clockwise* for the sense of a rotation are firmly entrenched. If a rotation when seen looking down the rotation axis toward the origin turns in the same sense as the hands of a clock, it is termed a clockwise rotation (cw), and if the rotation is in the opposite sense, it is counterclockwise (ccw) as illustrated by the sketches. This text follows the usual convention that counterclockwise rotations are positive.



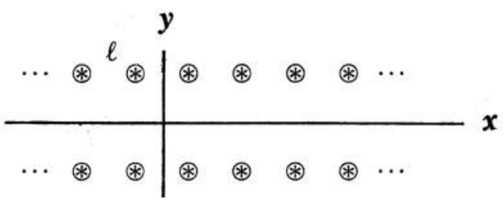


The sketch shows the effect of the operations **E**, **A**, and **B** on the triangle. The operations have left the appearance of the triangle unchanged, which is the essence of the concept of symmetry. Frank Wilczek (Nobel laureate in physics 2004) coined a pithy phrase to describe the connection between operations and symmetry: "change without change." With reference to the triangle example, we have made a change – an operation was performed on the triangle by rotating it – but the triangle still looks the same.



More generally, if an operation on an object leaves it unchanged, or *invariant*, the object must have a symmetry property. In the triangle example a 3-fold symmetry is revealed by rotation through the particular angles 0°, 120°, and 240°.

The symmetry of an equilateral triangle under certain rotations is an example of a *rotation* symmetry. There are many other examples of geometric symmetry. Consider the repeated pattern in the sketch, which could be a decorative frieze along the edge of a building. The two rows are parallel to the x-axis and are equidistant from the x-axis. The columns are all equally spaced along x by a distance ℓ, and the dots signify that the pattern extends far to the left and far to the right.

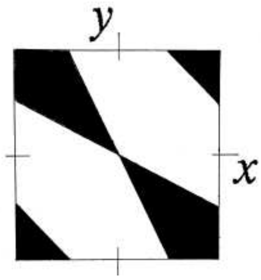


If the pattern is translated parallel to the x-axis by an integer multiple of ℓ, its appearance remains the same. This is an example of *translation* symmetry, important for the discussion of crystal lattices in Chapter 4.

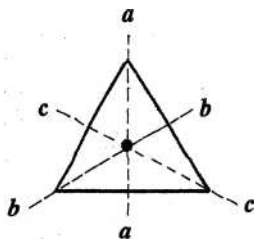
If the pattern is folded along the x-axis, the two rows coincide. Each row is a mirror image of the other, an example of *reflection* symmetry, also called *mirror* symmetry.

Symmetry is appealing and has long played a role in art and architecture, from ancient rock carvings to mosaics in ancient Rome to ephemeral foam patterns on coffee drinks.

For the black-and-white geometric pattern in the figure, the z-axis is normal to the page and passes through the origin. Rotations about z by 0° and 180° and reflections about the diagonals are symmetry operations. Rotations about z by 90° and 240° and reflections about the x- and y-axes are not symmetry operations.



The equilateral triangle has additional symmetries revealed by no longer requiring the triangle to lie in the x-y plane. The figure shows three new axes aa, bb, and cc. Each axis passes through an apex and is perpendicular to the opposite edge. It follows by geometry that the axes intersect at the geometric center of the triangle.



Suppose now that the triangle is "flipped" 180° about axis aa. The front becomes the back and vice versa; the appearance of the triangle is unchanged, so this is a symmetry operation on the triangle, and similarly for flips about axes bb and cc.

The three rotation operations in the plane and the three flip operations identify six symmetry operations for the equilateral triangle. These operations are easily demonstrated with a cardboard triangle.

Inversion symmetry is abstract and cannot be shown pictorially or demonstrated by a physical model. Space inversion reverses the signs of the coordinates so that x is replaced by $-x$, y by $-y$, and z by $-z$. These replacements are conveniently expressed by the symbol $\mapsto$, which means "maps to" or "is replaced by." Thus, inversion can be written $x \mapsto -x$, $y \mapsto -y$, and $z \mapsto -z$.

Consider a sphere of radius R, which can be described algebraically by the equation $x^2 + y^2 + z^2 = R^2$. Upon applying the space inversion operation to the coordinates, the equation is unchanged; the sphere is invariant under space inversion. We shall see important examples of inversion when symmetry and the quantum theory of atoms are discussed in Chapter 5.

But the use of symmetry in decorative arts and the description of geometric figures barely scratches the surface of its deep importance. Steven Weinberg (1933–2021, Nobel laureate in physics 1979) has written that symmetry is the "key to nature's secrets," which is why the application of symmetry principles to physical problems is the subject of this text.

1.2.2 Products of Operations

Consider again the set of three operations $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ from the triangle example discussed in Section 1.2.1. The *product* of two operations is the result of applying first one operation followed by a second. If, for example, $\mathbf{A}$ is applied first, followed by $\mathbf{B}$, the product is written symbolically as $\mathbf{BA}$. The operation on the right, here $\mathbf{A}$, is considered to be applied first. Note that although the product $\mathbf{BA}$ has the appearance of "multiplication" of $\mathbf{B}$ times $\mathbf{A}$, abstract group theory puts no restrictions on the method by which operations are actually combined. Some books on group theory use the term "multiplication" where we use "product." Such terms are symbolic only, with no reference to ordinary arithmetic.

In the example, $\mathbf{B}$ is applied to $\mathbf{A}$ "from the left." Alternatively, an operation can be applied "from the right" to give, in this case, $\mathbf{AB}$. These same ideas are also used with equations relating operations. Equations involving operations conform to the usual rule from algebra that both sides are to be treated equally. Consider, for example, the product of two operations $\mathbf{T_1}$ and $\mathbf{T_2}$ to give a third operation $\mathbf{T_3}$:

$$\mathbf{T_2T_1} = \mathbf{T_3}.$$

Now apply an operation $\mathbf{C}$ from the left; $\mathbf{C}$ must act on both sides of the relation to maintain the equality.

$$\mathbf{CT_2T_1} = \mathbf{CT_3}$$

Applying $\mathbf{C}$ from the right gives

$$\mathbf{T_2T_1C} = \mathbf{T_3}C.$$

The distinction between operations from the left and from the right is important. The reason is that for many group operations the order of combination makes a difference, unlike the multiplication of numbers or algebraic functions. If two operations $\mathbf{T_1}$ and $\mathbf{T_2}$ are combined, the two possible products $\mathbf{T_1T_2}$ and $\mathbf{T_2T_1}$ may not necessarily be equal. However, if $\mathbf{T_1T_2} = \mathbf{T_2T_1}$, then $\mathbf{T_1}$ and $\mathbf{T_2}$ are said to *commute*.

1.2.3 Product Tables

Rotation symmetry operations on the equilateral triangle are rotations through defined angles about defined axes, so it is easy to determine the product of any two operations. Consider, for example, the product $\mathbf{BA}$. First applying $\mathbf{A}$ produces an initial rotation of 120°. The second operation $\mathbf{B}$ causes an additional rotation through 240°, for a net result of 360° (equivalently 0°). This is the same result as using operation $\mathbf{E}$ alone, so the product is written

$$\mathbf{BA} = \mathbf{E}.$$

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Robert Kolenkow
Excerpt
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Table 1.1 Products of **E**, **A**, and **B**

	E	**A**	**B**
E	**EE** = **E**	**EA** = **A**	**EB** = **B**
A	**AE** = **A**	**AA** = **B**	**AB** = **E**
B	**BE** = **B**	**BA** = **E**	**BB** = **A**

Table 1.2 Products of **E**, **A**, and **B**

	E	**A**	**B**
E	**E**	**A**	**B**
A	**A**	**B**	**E**
B	**B**	**E**	**A**

The same reasoning can be used to evaluate all of the nine possible products of **E**, **A**, and **B**, being sure as a general rule to maintain the order of the operations. The products are conveniently displayed in the form of a *product table*, where an operation in the top horizontal row is applied first followed by an operation from the left-hand vertical column. For clarity in this first illustration, both the product and the net result are given in Table 1.1, but after this a table will show only net results, as in Table 1.2.

The tables show that $\mathbf{AA} = \mathbf{A}^2 = \mathbf{B}$; geometrically, two successive counterclockwise rotations by 120° give the same result as a single counterclockwise rotation by 240°. All the members of this set are powers of a single member **A**. The triangle rotation operations **E**, **A**, and **B** are *cyclic* because they can all be written as powers of **A**: $\mathbf{E} = \mathbf{A}^0$, $\mathbf{A} = \mathbf{A}^1$, and $\mathbf{B} = \mathbf{A}^2$.

Table 1.2 shows that in this particular example the operations $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ all commute with one another, for instance, $\mathbf{AB} = \mathbf{BA}$. The identity operation **E** always commutes with any operation **T** because $\mathbf{ET} = \mathbf{TE} = \mathbf{T}$.

1.2.4 The Inverse of an Operation

For any operation **T** there may be an *inverse* operation, symbolized $\mathbf{T}^{-1}$, that undoes the effect of **T** on the operand. Because the identity operation **E** always signifies no change, it follows that $\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{E}$. An operation always commutes with its inverse.

In the triangle example, **A** is a counterclockwise rotation through 120°, so one way to undo the effect of **A** is by a further counterclockwise rotation through an additional 240°, to give a net rotation of 360° = 0°. In the set $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ the inverse of

A is identified as $\mathbf{A}^{-1} = \mathbf{B}$. By similar reasoning, $\mathbf{B}^{-1} = \mathbf{A}$. These results can also be read from Table 1.2. The entries $\mathbf{AB} = \mathbf{BA} = \mathbf{E}$ show, for example, that $\mathbf{B} = \mathbf{A}^{-1}$.

Another operation that undoes the effect of **A** is to rotate clockwise through 120° to bring the triangle back to the starting point. This clockwise rotation is equivalent to a counterclockwise rotation through −120°. This is a new operation and not a member of the operations **E**, **A**, and **B**, which are defined here only for counterclockwise rotations.

Here is an example involving inverses and a product table. Consider the set $\{\mathbf{E}, \mathbf{A}\}$ with the following partial product tables:

	E	**A**
E	**E**	**A**
A	**A**	$\mathbf{A}^2$

What is the unidentified member $\mathbf{A}^2$? Try $\mathbf{A}^2 = \mathbf{A}$. Multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\begin{aligned}
\mathbf{A}^2 &= \mathbf{A} \\
\mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}\mathbf{A} \\
(\mathbf{A}^{-1}\mathbf{A})\mathbf{A} &= \mathbf{E} \\
\mathbf{E}\mathbf{A} &= \mathbf{E} \\
\mathbf{A} &= \mathbf{E}
\end{aligned}$$

The result $\mathbf{A} = \mathbf{E}$ gives the dull and useless Table 1.3.

The alternative possibility $\mathbf{A}^2 = \mathbf{E}$ gives the more useful product Table 1.4 that has two distinctly different members.

In the product table for a set of operations, a given symmetry operation appears only once in each column as seen in the example. As a proof consider a set of distinctly different symmetry operations **A**, **B**, **C**, and **D**. Suppose that **A** occurs twice in the column headed by **B**, so that $\mathbf{BC} = \mathbf{A}$ and $\mathbf{BD} = \mathbf{A}$. Then $\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}$ and $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}$ so that $\mathbf{C} = \mathbf{D}$, a contradiction because the operations are assumed to be different. Similarly, each operation occurs only once in a given row.

Table 1.3 $\mathbf{A} = \mathbf{E}$

	E	**E**
E	**E**	**E**
E	**E**	**E**

Table 1.4 $\mathbf{A}^2 = \mathbf{E}$

	E	**A**
E	**E**	**A**
A	**A**	**E**

1.3 What Is a Group?

With a solid foundation on the nature of operations, their products, and their inverses, it is time to take up the heart of the matter: the definition of a group. The definition is summarized in the following five axioms (i) to (v). They may seem a little dry, but they are needed because if a set of operations can be shown to form a group, a raft of useful theorems are then immediately applicable.

To illustrate the axioms, we shall show that the set of triangle rotation operations $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ form a group.

(i) A group consists of a set of operations called *members* of the group. We shall show that the set $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ are members of a group.

(ii) The product of any two members of a group is also a member of the group; products do not take us to new operations outside the set of group members. Table 1.2 shows that the products of **E**, **A**, and **B** are all members of the same set. Contrariwise, clockwise rotations of the triangle do not appear in Table 1.2 and are therefore not members of this group.

(iii) The group contains an identity member **E** that produces no change when combined with any group member. Table 1.2 for the triangle rotations show that $\mathbf{EE} = \mathbf{E}$, $\mathbf{EA} = \mathbf{AE} = \mathbf{A}$, and $\mathbf{EB} = \mathbf{BE} = \mathbf{B}$, showing that the notation is justified; **E** is truly the identity member in the set.

(iv) For every member **T** of a group, there is also a member $\mathbf{T}^{-1}$ in the group that is the inverse of **T**, such that $\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{E}$. As shown in Section 1.2.4 and also in Table 1.2, $\mathbf{E}^{-1} = \mathbf{E}$, $\mathbf{A}^{-1} = \mathbf{B}$, and $\mathbf{B}^{-1} = \mathbf{A}$.

(v) An additional axiom is that the products of operations are *associative* so that $\mathbf{T_1}(\mathbf{T_2}\mathbf{T_3}) = (\mathbf{T_1}\mathbf{T_2})\mathbf{T_3}$, where the products in parentheses are evaluated first, then combined with the remaining operation. This axiom will be satisfied automatically by the operations in applications.

Let $\boldsymbol{\Gamma}$ be the symbol for the group $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$. The number of members in a group is called the *order* of the group: $\boldsymbol{\Gamma}$ is of order 3.

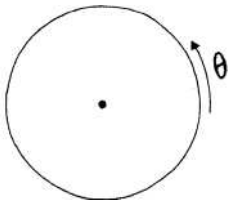
Note that **E** is always a member of any group and satisfies the group definition axioms. Therefore **E** is itself a group (of order 1). If a subset of group members are themselves a group, the subset is called a *subgroup*. **E** is a trivial subgroup of every group. The whole group itself is also a trivial subgroup of the group.

In the group $\mathbf{\Gamma}$, the set $\{\mathbf{E}, \mathbf{A}\}$ is not a subgroup, because the product $\mathbf{AA} = \mathbf{B}$, an operation not included in the set. In a subgroup, just as in a group, the product of two operations in the subgroup must also be a member of the subgroup.

The product table for a set of operations can be checked to see whether the group axioms are satisfied. The product table tells all.

1.3.1 Discrete and Continuous Groups

Groups with a finite (countable) number of members are called *discrete* or *finite* groups. The triangle rotation group $\mathbf{\Gamma} = \{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ has a finite number of members and is an example of a discrete group.



Consider now a flat circular disk with a perpendicular axis through its center, as suggested by the sketch. Rotation of the disk by an arbitrary counterclockwise angle θ leaves the disk invariant, so this leads us to suspect that there is a group involving rotations. The rotations form a group: rotation by 0° is the identity, two successive rotations by θ_1 and θ_2 give the same result as a single rotation by $\theta_1 + \theta_2$, and to every rotation θ there corresponds an inverse rotation $360° - \theta$.

Because θ can be any angle, this group has an "infinite" (uncountable) number of members; it is an example of a *continuous group*. A continuous group depends on a continuous parameter, in this example the angle θ. Continuous groups are important in physics, for example, in the quantum-mechanical wave function of a hydrogen atom, which depends on two continuous parameters: the polar angle θ and the azimuthal angle ϕ.

1.4 Examples of Groups

1.4.1 Abelian Groups

A group in which all of the members commute is called an *Abelian* group, after the Norwegian mathematician Niels Henrik Abel (1802–29). The triangle rotation group composed of the set $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ is an Abelian group. This group is also a cyclic group and can be written as $\{\mathbf{E}, \mathbf{A}, \mathbf{A}^2\}$ as shown in Section 1.2.3.

All groups of order less than 6 are Abelian.

1.4.2 The 32 Group

Table 1.5 is the product table for a group of order 6, a popular example in textbooks on group theory. It is termed the **32** ("three-two") group.

Table 1.5 The **32** group (order 6)

	E	**A**	**B**	**C**	**D**	**F**
E	**E**	**A**	**B**	**C**	**D**	**F**
A	**A**	**E**	**F**	**D**	**C**	**B**
B	**B**	**D**	**E**	**F**	**A**	**C**
C	**C**	**F**	**D**	**E**	**B**	**A**
D	**D**	**B**	**C**	**A**	**F**	**E**
F	**F**	**C**	**A**	**B**	**E**	**D**

The product table shows that the group axioms are satisfied:

(i) Only members from the set appear in the product table.

(ii) The product of two members is a member of the set.

(iii) There is an identity member identified as **E** that obeys the properties of the identity operation such as $\mathbf{EA} = \mathbf{AE} = \mathbf{A}$.

(iv) Every member of a group has an inverse in the set as shown by products such as $\mathbf{DF} = \mathbf{E}$ so that $\mathbf{F} = \mathbf{D}^{-1}$.

Members of a given group may or may not commute. For example, $\mathbf{AB} = \mathbf{F}$ and $\mathbf{BA} = \mathbf{D}$. **A** and **B** do not commute, so **32** is not an Abelian group. It is the smallest group that is nonAbelian, accounting for its popularity as a teaching tool.

Table 1.5 shows that the **32** group has three nontrivial subgroups of order 2, namely $\{\mathbf{E}, \mathbf{A}\}$, $\{\mathbf{E}, \mathbf{B}\}$, and $\{\mathbf{E}, \mathbf{C}\}$ and also a subgroup of order 3 $\{\mathbf{E}, \mathbf{D}, \mathbf{F}\}$, but no others. A theorem in group theory states that for a group of order n each of its subgroups has an order that is a factor of n. The example of the **32** group demonstrates this theorem because $6 = 2 \cdot 3$ for the nontrivial subgroups of orders 2 and 3. $6 = 6 \cdot 1$ is satisfied by the trivial subgroup $\{\mathbf{E}\}$ of order 1 and by the group itself of order 6.

It follows from this theorem that if the order of a group is a prime number, the group has no nontrivial subgroups and must therefore be a cyclic group. The group $\mathbf{\Gamma}$ of order 3 is an example.

1.4.3 The Permutation (Symmetric) Group

This section discusses the apparently different example of the *permutation group*. Here is the permutation group of order 6.

$$\underset{\mathbf{P}_1}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}} \quad \underset{\mathbf{P}_2}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}} \quad \underset{\mathbf{P}_3}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}} \quad \underset{\mathbf{P}_4}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}} \quad \underset{\mathbf{P}_5}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}} \quad \underset{\mathbf{P}_6}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}$$