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Part I

Normal Mode Instabilities

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1
Preliminaries

If you would understand anything, observe its beginning.
– Aristotle

Oceanographers think of ocean circulation in terms of the "global conveyer belt," in which cold polar waters sink and then circulate around the ocean basins, eventually being warmed in the tropics. But the truth is that this larger-scale circulation has a typical speed of only a few cm s^{-1}, and it is generally accompanied by variable currents many times faster (1 m s^{-1} is not uncommon), fluctuating on periods ranging from months down to seconds.

The largest such variations are the majestic mesoscale eddies that spin off strong currents like the Gulf Stream (Figure 1.1). Today, much research is focused on the

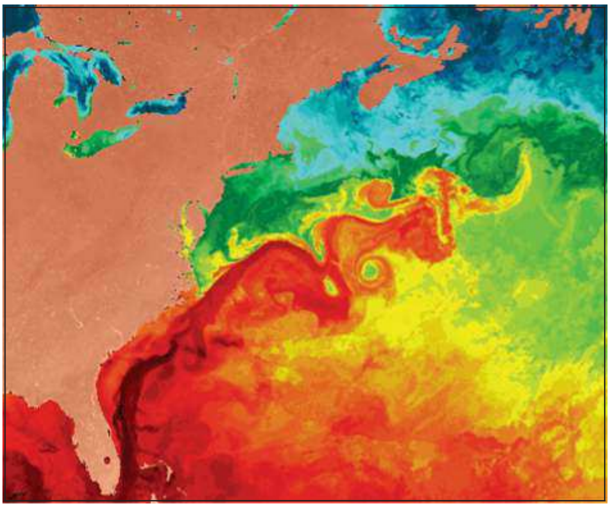

Figure 1.1 Instability of the Gulf Stream shown in a satellite image. Colors represent sea surface temperature. The darkest red represents warm water flowing northeast along the east coast of the United States. After departing from the coast at Cape Hatteras, the current becomes unstable and breaks down into turbulent eddies. (Image courtesy of the U.S. National Oceanic and Atmospheric Administration, hereafter NOAA.)

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next size smaller: the submesoscale eddies. Smaller than this are the gravity waves and, at the smallest scale, three-dimensional turbulence.

One must measure for a year or more in order to average out these various fluctuations and discern the mean "conveyer belt" current. But to think of the variations as something we can average away is to fool ourselves, for it is largely the oscillations that govern the conveyer belt. We can't understand one without the other.

One way to understand such a chaotic profusion of motions is to ask what would happen if, at some initial instant, the ocean was calm, with steady, orderly currents. Would the oscillations develop spontaneously? If so, how? This thought experiment is the essence of instability theory.

For example, Figure 1.1 shows eddies forming in the Gulf Stream. No human mathematician could solve the equations that describe these intricate motions; but, using the methods of linear perturbation theory, we can not only predict their length and time scales but also understand quite a lot about what causes them. The trick is to imagine a fictitious Gulf Stream that is straight and eddy-free, then study what happens in the very first few moments – after the current begins to buckle but before it grows so complex as to be mathematically intractable.

We can think of atmospheric motions in the same way. Imagine a fictitious atmosphere where the winds are purely zonal – mid-latitude westerlies, jet streams, and polar easterlies, all blowing straight in the east-west direction only. It turns out that those winds would be unstable, and as a result would break up into the large vortical structures we know as synoptic weather systems (Figure 1.2). To calculate the details requires a supercomputer, but we can understand the basic mechanics and predict the dominant length and time scales (a few thousand kilometers, a few days) quite easily.

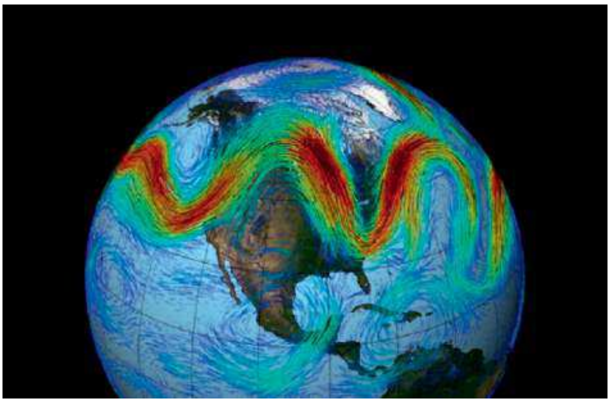

Figure 1.2 The atmospheric jet stream: speed (red = fast) and direction (streaks), showing baroclinic instability. (Visualization courtesy of the US National Aeronautics and Space Administration, hereafter NASA).

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The Earth's mantle provides a third example. Suppose the mantle were perfectly motionless. Heating from the radioactive core would lead to the growth of convection cells, exactly as we see reflected in the slow drift of the continents and the attendant seismic and volcanic activity.

In this book we will study instabilities on scales from centimeter to global, controlled mainly by gravity, sheared winds and currents, and the Earth's rotation. While our main focus is the Earth, analogous phenomena are found in atmospheres and magnetospheres of other planets, stellar interiors, and interstellar plasma flows.

1.1 What Is Instability?

Suppose that the emergency brake in your car doesn't work, and you have to park somewhere in hilly country. Where can you park so that your car doesn't roll away (Figure 1.3)? We hope you would park at point (a), the bottom of a valley. But what about point (b), the top of a hill? You could park there in theory, but you would have to park at *exactly* the right spot, and even then any little disturbance would cause your car to roll away.

In mathematical terms, we say that both points (a) and (b) are equilibrium states,[1] i.e., states at which the system can remain steady in time. The difference between (a) and (b) is in what happens when the system is displaced slightly from equilibrium. If you park at the bottom of the valley (a), and if something then pushes the car slightly to the left or the right, gravity will pull it back toward its original location. The car will rock back and forth and eventually come to rest due to friction. In contrast, if you park at the top of the hill (b) and the car is moved even slightly, gravity pulls it further from the equilibrium point. The further the car travels, the steeper the slope and the stronger the pull of gravity. Goodbye car! We say that equilibrium (a) is stable, while (b) is unstable.

The equations that describe geophysical fluid systems are in general far too complicated to solve analytically. One way to get around this problem is to look for equilibria, i.e., solutions that are valid when all time derivatives are set to zero. Flows are often found to be close to such equilibria. For example, the surface of

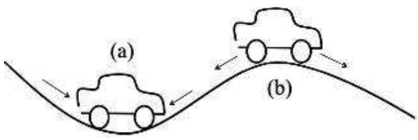

Figure 1.3 No brakes! Where would you park? Arrows show the gravitational force that acts when the car is displaced slightly from equilibrium.

[1] Highlighted text is used as an extra level of emphasis for important concepts.

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a lake is in equilibrium if it is horizontal. Although this is never exactly true, it is pretty close on average.

Once we have identified an equilibrium state, the next step is to determine its stability. If the equilibrium is stable, disturbances will often have the form of oscillations (e.g., the car in Figure 1.3a), or waves. If the equilibrium is unstable, then small disturbances grow exponentially. Instabilities will be our focus here, though we will find it useful to examine wave phenomena as well.

1.1.1 The Cycle of Instability

Because unstable systems are by their nature ephemeral, you might reasonably wonder why we ever observe them. It is much more usual to see systems close to stable equilibria. For example, the surface of a lake is never perfectly horizontal, but it's usually pretty close, because the horizontal equilibrium state is stable.

But a sufficiently strong wind destabilizes that horizontal equilibrium state, and waves grow as a result.[2] If the waves grow large enough, they fall prey to a second kind of instability as the crests roll over and break (related to convective instability; Chapter 2). The surface then relaxes toward the horizontal state until a new set of waves emerges. Eventually the wind dies down and the horizontal state is once again stable.

The oceans and atmosphere are almost always turbulent, and this cycle of instability is the reason. Forcing by wind, sun, gravity, and planetary rotation tends to push the system toward unstable states. Instability and turbulence then act to relax the system back toward stability. This cyclic instability regime is discussed further in section 12.3.

1.2 Goals

Our exploration of instability will have three main goals.

(i) **Mechanisms:** We aim to understand, on an intuitive level, the basic physical processes that generate instability. In the car example, we've seen how motion away from equilibrium alters the effect of gravity (arrows in Figure 1.3), resulting in oscillations or instability. Geophysical examples will take a bit more work to understand, but we'll do it.

(ii) **Rules of thumb:** We would like to be able to predict the stability or instability of a system quickly with minimal math. In the car example, we are able to predict whether an equilibrium point will be stable or unstable without knowing the details of the shape of the hill or valley. All we need to know

[2] The process is similar to shear instability, covered in Chapters 3–5.

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is whether the equilibrium is a maximum or a minimum of elevation, i.e., whether the curvature at that point is negative or positive.

We can invent similar rules for most types of geophysical flow instability. These allow us to estimate not only the likelihood of instability, but also the spatial and temporal scales on which it will grow. These can help us identify the particular mechanism through which a geophysical flow becomes unstable. For example, the Gulf Stream eddies shown in Figure 1.1 could be due to different instabilities (which you will learn about later). By comparing their observed length scales, and the time they take to grow, with rules of thumb based on various known instability types, we can take a first guess as to the mechanism.

(iii) **Numerical solution methods:** Sometimes a rule of thumb is not enough. We want to determine quantitative details of an instability, perhaps in a situation where many physical factors interact. In that case we may have to solve a nontrivial set of differential equations. Many advanced analytical methods are available, but in this book we will focus on numerical methods. Since the 1980s, computers have had the capacity to do something unprecedented: *to solve a differential equation whose coefficients are defined using observational measurements*. That capability is now in use in the analysis of oceanographic and atmospheric observation.

1.3 Tools

Below are three topics we'll expect you to have some familiarity with. Under each topic is listed one or more things that you should be able to do.

(i) **Calculus**:

- Solve this boundary value problem:

$$y'' = -y; \quad y(0) = y(\pi) = 0. \tag{1.1}$$

- Derive this Taylor series approximation:

$$\frac{1}{1+x} \approx 1 - x + x^2, \quad \text{for } |x| \ll 1.$$

- Understand the meaning of (though not necessarily solve) a partial differential equation, e.g.,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \pi}{\partial x}.$$

- Define the Dirac delta function.

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(ii) **Linear algebra**:

- Compute the eigenvalues of a 2×2 matrix.

(iii) **Programming**: Homework will be done using the Matlab programming environment[3] or something equivalent.[4] You do not have to be an expert; you will learn as you go. But if you've never used the software at all it would be worth familiarizing yourself with the basic syntax. Try the following:

- Write a function, and a script that calls it.
- Define a matrix and compute its eigenvalues.
- Make a line plot and label the axes.
- Make an image plot.

1.4 Numerical Solution of a Boundary Value Problem

The basic geophysical flow instabilities are analyzed as solutions of two-point boundary value problems. In this section we'll define this class of problems and introduce a simple method for solving them.

1.4.1 Defining the Problem

Let $f(x)$ be the solution to a second-order ordinary differential equation with independent variable x. Complete specification of f requires two pieces of information in addition to the equation itself. These can be either

- values of f and its first-derivative at some initial point, which we'll label as zero, i.e., $f(0)$ and $f'(0)$, or
- values of f at two points, say $f(0)$ and $f(L)$.

The first case is called an initial value problem; the second is called a boundary value problem.

A critical difference between these two classes of problem is that the first generally has a solution while the second generally does not. Here's a simple example:

[3] Many universities make the Matlab software available free to students.

[4] Python is another programming environment that we recommend. It is freely available at www.python.org. The commands you need here are found in two packages that will be used over and over. Most of the numerical mathematics and matrix operations come from the `numpy` package, whereas the plotting commands are from the `matplotlib` package. You should start your Python scripts with the following lines:

```
import numpy as np
import matplotlib as plt
```

to load these packages and give them the short cuts `np` and `plt`, respectively. Plotting a line can then be done with the command `plt.plot`, and finding eigenvalues can be accomplished with `np.eig`.

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$$f'' = -k^2 f. \tag{1.2}$$

The general solution is

$$f = A\cos kx + B\sin kx, \tag{1.3}$$

where A and B are constants to be determined. Consider first the initial value problem. Suppose we have initial conditions $f(0) = 0$ and $f'(0) = 1$. The solution is then (1.3) with $A = 0$ and $B = 1/k$ (try it). Note that this solution works for *any* value of k.

Now, consider the boundary value problem with conditions

$$f(0) = 0; \quad f(L) = 0. \tag{1.4}$$

The first condition is satisfied if $A = 0$, but the second can then be satisfied only if $k = \pm n\pi/L$, where n is any integer. These special values of k are called the eigenvalues of the problem, and unless k is equal to one of those eigenvalues, the problem has no solution. We also call (1.2–1.4) a differential eigenvalue problem. It is analogous to the more familiar matrix eigenvalue problem, and can in fact be solved numerically using the same methods.

Here's how it works. Suppose that

- $\vec{x}$ is a list of possible values of x arranged as a vector;
- $\vec{f}$ and $\vec{f}^{(2)}$ are vectors composed of the corresponding values of f and its second-derivative, respectively;
- D is a matrix such that $\mathsf{D}\vec{f} = \vec{f}^{(2)}$.

We can now write (1.2) as

$$\mathsf{D}\vec{f} = -k^2\vec{f}, \tag{1.5}$$

which is a standard matrix eigenvalue problem with eigenvalue $-k^2$. Because the matrix eigenvalue problem can be easily solved using standard numerical routines (e.g., the Matlab function `eig`[5]), this approach suggests a convenient way to solve the differential eigenvalue problem. But first we have to identify this matrix D that transforms a vector into its second-derivative.

1.4.2 Discretization and the Derivative Matrix

We discretize the independent variable x by choosing a vector of values:

$$x_i = x_0 + i\Delta, \quad \text{where } i = 0, 1, 2, \ldots, N+1.$$

[5] `Blue text` indicates Matlab syntax. We give coding examples in Matlab, assuming that readers preferring other software environments will substitute the equivalent expressions.

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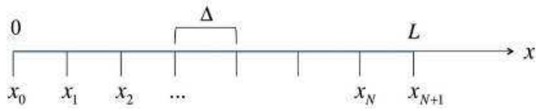

Figure 1.4 Discretizing the x axis.

The first and last values correspond to the boundaries, say $x_0 = 0$ and $x_{N+1} = L$ (Figure 1.4). This requires that

$$\Delta = L/(N+1).$$

Note that the x_i are evenly spaced. This restriction is not necessary, but it simplifies the math. We can now discretize the solution f,

$$f_i = f(x_i),$$

and the kth derivative

$$f_i^{(k)} = \left.\frac{d^k f}{dx^k}\right|_{x=x_i}.$$

The finite difference approximation to the derivative $f^{(k)}$ is a weighted sum of f_i values at neighboring points. A well-known example is:

$$f_i' = \frac{f_{i+1} - f_i}{\Delta}, \tag{1.6}$$

which approximates the first-derivative to arbitrary accuracy as $\Delta \to 0$. In general

$$\boxed{f_i^{(k)} = \sum_{j=j_1}^{j_2} A_j^{(k)} f_{i+j}.}$$

The range of the summation, $j = j_1, \ldots, j_2$, is called the stencil. For example, in (1.6), $k = 1$, $j_1 = 0$, and $j_2 = 1$, and the weights are $A_0^{(1)} = -1/\Delta$ and $A_1^{(1)} = 1/\Delta$.

The weights are computed by means of a Taylor series expansion of f about x_i:

$$\boxed{f_{i+j} = f(x_i + j\Delta) = f_i + j\Delta f_i^{(1)} + \frac{1}{2}(j\Delta)^2 f_i^{(2)} + \ldots + \frac{1}{k!}(j\Delta)^k f_i^{(k)}.} \tag{1.7}$$

For example, suppose we want to approximate the first-derivative using the three-point stencil $j = -1, 0, 1$:

$$\tilde{f}_i' = A f_{i-1} + B f_i + C f_{i+1},$$

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