The Black–Scholes–Merton Model as an Idealization of Discrete-Time Economies

This book examines whether continuous-time models in frictionless financial economies can be well approximated by discrete-time models. It specifically looks to answer the question: In what sense and to what extent does the famous Black–Scholes–Merton (BSM) continuous-time model of financial markets idealize more realistic discrete-time models of those markets? While it is well known that the BSM model is an idealization of discrete-time economies where the stock price process is driven by a binomial random walk, it is less known that the BSM model idealizes discrete-time economies whose stock price process is driven by more general random walks. After recounting the foundations of discrete- and continuous-time models of financial markets, David M. Kreps develops the general theory of convergence of discrete-time models to their continuous-time limits, both in mathematical and economic terms. The exposition lowers the entry barriers to the literature of financial mathematics for less-technical readers, while providing a fuller understanding of the connections between BSM and nearby discrete economies.

David M. Kreps is the Adams Distinguished Professor of Management, Emeritus, at the Stanford Graduate School of Business. He has been honored with many awards, including the John Bates Clark Medal by the American Economic Association in 1989 and the John J. Carty Award for the Advancement of Science by the National Academy of Sciences in 2018.

Econometric Society Monograph Series

Editors:

Andrea Prat, Columbia University
Stéphane Bonhomme, University of Chicago

The Econometric Society is an international society for the advancement of economic theory in relation to statistics and mathematics. The Econometric Society Monograph series is designed to promote the publication of original research contributions of high quality in mathematical economics and theoretical and applied econometrics.

Books in the Series

W. Thomson, *How to Divide When There Isn't Enough: From Aristotle, the Talmud, and Maimonides to the Axiomatics of Resource Allocation*, 2019
O. Compte & A. Postlewaite, *Ignorance and Uncertainty*, 2018
I. Molchanov & F. Molinari, *Random Sets in Econometrics*, 2018
B. Honoré, A. Pakes, M. Piazzesi, & L. Samuelson (eds.), *Advances in Economics and Econometrics: Eleventh World Congress, Vols. I & II*, 2017
S. Maurer, *On the Shoulders of Giants: Colleagues Remember Suzanne Scotchmer's Contributions to Economics*, 2017
C. P. Chambers & F. Echenique, *Revealed Preference Theory*, 2016
J.-F. Mertens, S. Sorins, & S. Samir, *Repeated Games*, 2015
C. Hsiao, *Analysis of Panel Data: 3rd edn*, 2014
C. Cameron & P. Trivedi, *Regression Analysis of Count Data, 2nd edn*, 2013
A. Harvey, *Dynamic Models for Volatility and Heavy Tails, with Applications to Financial and Economic Time Series*, 2013
D. Acemoglu, M. Areilano, & E. Dekel (eds.), *Advances in Economics and Econometrics: Theory and Applications, Tenth World Congress, Vols. I, II, & III*, 2013
M. Fleurbaey & F. Maniquet, *A Theory of Fairness and Social Justice*, 2011
R. Vohra, *Mechanism Design: A Linear Programming Approach*, 2011
K. Samphantharak & R. Townsend, *Households as Corporate Firms: An Analysis of Household Finance Using Integrated Household Surveys and Corporate Financial Accounting*, 2009
I. Gilboa, *Theory of Decision under Uncertainty*, 2009
F. Vega-Redondo, *Complex Networks*, 2007
R. Blundell, W. Newey, & T. Persson, (eds.), *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress, Vols. I, II, & III*, 2006
J. Roemer, *Democracy, Education, and Equality*, 2006
C. Blackorby, W. Bossert, & D. Donaldson, *Population Issues in Social Choice Theory, Welfare Economics and Ethics*, 2005
R. Koenker, *Quantile Regression*, 2005
C. Hsiao, *Analysis of Panel Data, 2nd ed.*, 2003
M. Dewatripont, L. P. Hausen, & S. J. Turnovsky (eds.), *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress, Vols. I, II, & III*, 2003
E. Ghysels, N. Swanson, & M. Watson (eds.), *Essays in Econometrics: Collected Papers of Clive W. J. Granger, Vols. I & II*, 2001
S. Strøm (ed.), *Econometrics and Economic Theory in the 20th Century: The Ragnar Frisch Centennial Symposium*, 1999
A. C. Cameron & P. K. Trivedi, *Regression Analysis of Count-Data*, 1998
D. Jacobs, E. Kalai, & M. Kamien (eds.), *Frontiers of Research in Economic Theory: The Nancy L. Schwartz Memorial Lectures*, 1998

Continued on page following the index

The Black–Scholes–Merton Model as an Idealization of Discrete-Time Economies

David M. Kreps
Stanford University, California

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108486361
DOI: 10.1017/9781108626903

First published 2019

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-48636-1 Hardback
ISBN 978-1-108-70765-7 Paperback

Contents

Preface

I began this monograph (which, at the time, was a nascent paper) with the objective of understanding how and how well continuous-time models of economic phenomena – and in particular models that employ Brownian motion – compare with "near by" discrete-time models. We know that the comparisons can sometimes be strained; see, for instance, Fudenberg and Levine (2009) and Sadzik and Stacchetti (2015). So, it seemed to me, a general theory connecting the two types of models ought to be available.

I still believe that such a general theory is available. But I quickly discovered that it is difficult, and so to build my intuition, I did what any faculty member in a school of management would do: I began to look at "case studies." In particular, I began with the connections between discrete- and continuous-time models of financial markets and, in particular, the connections between the famous Black–Scholes–Merton[1] theory of pricing contingent claims by arbitrage and discrete-time models – the seminal reference is Cox, Ross, and Rubinstein (1979) – that, in the limit, "converge" to the Black–Scholes–Merton (BSM, for short) model. The scare quotes around "converge" signify that the point of the exercise is to understand what convergence means in this context and, in particular, to understand how general is this convergence.

In the course of studying these issues, I first discovered that, in some regards, BSM is the economic "limit" of more discrete-time models than I had originally thought – developing a precise statement of what this means is one of the main points of this monograph – and then that, while this fact was not known by me and, seemingly, is not well known by many of colleagues in the field of Finance, it is well known to scholars – primarily mathematicians who specialize in probability – who are members of the community of so-called Financial Mathematicians. I believe that what is known to this community of scholars should be well-known more broadly to financial economists, and so I embarked on writing an exposition of those ideas *in the simplest possible settings* and in a single, unified presentation, so they would be accessible to mainstream financial

[1] It is common in the literature to refer to the Black–Scholes theory, the Black–Scholes Option Pricing Model, and the Black–Scholes Formula. But the history of these ideas supports giving equal credit to Robert Merton. Hence, in this monograph, I refer to the Black–Scholes–Merton model and theory.

economists. This monograph is the result.

To be clear, "in the simplest possible setting" does not mean "in a simple setting." The limit is the BSM model of an economy, which is mathematically a complex thing. And, to give precise limiting results, I must employ some mathematically sophisticated concepts and results, among which are weak convergence of probabilities on spaces of functions, the classic functional central limit theorem of Donsker, and the Skorohod Representation Theorem. Many mainstream financial economists (and economists more generally) will not find this an easy read, although as each of these tools is introduced, I provide analogies to results on the real line that most readers will know.

At the same time, this is not an archive of everything known on these issues. To make this more accessible, I do not give general results, nor do I pose the results in the mathematically most pleasing settings. (In particular, experts will be dismayed that I conduct business in $C[0,1]$ rather than in $D[0,1]$.) My hope is that the exposition will bring less-technical readers along to (at least) a good understanding of the connections between BSM and near by discrete-time economies.

Website

While I hope and expect that this monograph contains only a few errata, it is inevitable that there are some. I may have missed pertinent references, about which readers will want to know. There are several questions and conjectures left open, which (I hope) I or others will resolve. And, of course, work on the topics discussed here is ongoing. For these reasons, a public website has been created at the URL *discrete2continuous.stanford.edu* at which you can find supplementary materials of these sorts. I recommend visiting this website, to see what is posted there, as you read this book. (If you discover errors or typos, or if you wish to suggest additional references, and in particular if you can resolve some of the questions I've left open, please email me at kreps@stanford.edu.)

Acknowledgements

While preparing this manuscript, I benefited from both technical assistance and comments by a number of colleagues. Ayman Hindy and Chi-fu Huang deserve special mention, as they provided both technical assistance and comments, and also kept me working on this with questions that I at first couldn't answer (and some that I still can't). Dimitri Vayanos read very carefully through the entire manuscript and made many sug-

gestions that greatly improved the final version. Walter Schachermayer was both generous and invaluable on technical aspects, most significantly concerning Proposition 5.1*b*, but concerning many other matters, as well. Assistance as well from David Aldous, Rama Cont, Darrell Duffie, Hans Föllmer, Michael Harrison, Friedrich Hubalek, Jean-Luc Prigent, and Ruth Williams is gratefully acknowledged. Two excellent anonymous reviewers provided further assistance.

Jeff Ely was initially the Econometric Society Monograph Series editor handling this book; midstream, Andrea Pratt took over. I am grateful to both of them for the expeditious handling of the process. Karen Maloney was the acquisition editor for Cambridge University Press; she, and her editorial assistant Rachel Blaifeder, had to deal with an author who has a well deserved reputation for intransigence when it comes to book production; they did so with grace and patience. Although this book is not within his normal purview in terms of subject matter, David Tranah of CUP was very helpful concerning some the technical details related to production. Adam Hooper did an excellent job coordinating production, and copyeditor Anne Valentine exhibited a combination of flexibility to the author's desires where it made sense and strength in insisting on flexibility from the author when that was required. The book is significantly improved in style and readability through her efforts.

Dedication

The economic ideas in this monograph are due to a number of authors, and acknowledgements are given as appropriate. But the basic insight about how these questions can be approached is due to Stephen A. Ross, whose untimely death removed from financial economics one of the giants of the field. I dedicate this monograph to his memory.