1
Outline, Notation, Preliminaries

1.1 General Context

Numerical simulations of relativistic nonequilibrium fluid dynamics[1] have had enormous success in describing, explaining, and predicting experimental data from relativistic nuclear collisions. At the same time, nuclear collision experiments have had a tremendous importance in pushing the development of relativistic fluid dynamics out of equilibrium, both on a formal level such as the first-principles derivation of the fluid dynamics equations of motion and on the practical level through the development of algorithms for obtaining numerical solutions. This interplay between theory and experiment has led to the creation of a vibrant subfield of relativistic nonequilibrium (or viscous) fluid dynamics which unites research from traditionally separate disciplines such as string theory, classical gravity, computational physics, nuclear physics, and high-energy physics. While the early "gold-rush" years aimed at constraining the shear viscosity of quantum chromodynamics (QCD) may be coming to an end, new, previously unexpected avenues have opened up in the past 10 years, such as the duality between fluids and gravity, the applicability of fluid dynamics to small systems below the femtoscale, the role of thermal fluctuations in relativistic systems, fluid dynamics in the presence of anomalies, anisotropic hydrodynamics, an action formulation for dissipative fluid dynamics, relativistic magneto-hydrodynamics, and the role of nonhydrodynamic modes. It is probably fair to say that 10 years ago, only very few people expected such rich and novel physics to emerge from the old discipline of hydrodynamics! At the time of writing, the research in relativistic viscous fluid dynamics is alive and well, with vibrant new ideas continuing to be proposed and new experimental data from the Relativistic Heavy-Ion Collider (RHIC) as well as the Large Hadron Collider (LHC) continuing to stream in.

[1] As is common in the literature, we will use the terms "fluid dynamics" and "hydrodynamics" synonymously.

This wealth of experimental data is key to confirm or rule out theory predictions, and, sometimes, to challenge the relativistic hydrodynamics community, as has happened, for instance, through the discovery of flow-like signals in proton–proton collisions at the LHC. The borders between the traditionally separate high-energy physics and high-energy nuclear physics communities, never solid to begin with, have now started to disintegrate completely, with all of the present LHC experiments (ALICE, ATLAS, CMS, and LHCb) having working groups directly or indirectly aimed at studying the properties of relativistic viscous fluids. Recently, gravitational wave observations from LIGO have added to the treasure trove of data by providing measurements of black-hole nonhydrodynamic modes as well as neutron star mergers, which likely will play a key role in calibrating future relativistic viscous fluid dynamics simulations of compact stars.

The influx of new experimental capabilities and manpower is a welcome addition to the field, which continues to grow and strengthen, with no obvious limit in sight. The future of relativistic fluid dynamics looks bright, indeed!

1.2 Timeline of Major Events

The current formulation of relativistic viscous fluid dynamics did not appear out of nowhere. The following timeline summarizes some of the major events (heavily biased by personal opinion!) that played a vital role in the development of the field.

- Pre-1950s: Work on relativistic equations of motion for viscous fluids by Maxwell and Cattaneo [1, 2].
- 1960s–1970s: Work on relativistic equations of motion for viscous fluids by Müller, Israel, and Stewart [3–5].
- 1982: Analytic fluid modeling of heavy-ion collision by Bjorken [6].
- 1980s–1990s: First theoretical calculations for transport coefficients in gauge theories [7–9].
- 1990s–early 2000s: Relativistic ideal fluid modeling of heavy-ion collisions by multiple groups [10–16], including predictions for observables, in particular for the magnitude of the so-called elliptic flow.
- 2000: Calculation of shear viscosity in gauge theories to leading order in weak coupling by Arnold, Moore, and Yaffe [17].
- 2001: Following a breakthrough discovery in string theory [18], calculation of the shear viscosity for a gauge theory to leading order in strong coupling by Policastro, Son, and Starinets [19].
- 2001: The Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) starts operation.
- 2003: Analytic calculation indicating a strong sensitivity of elliptic flow to changes in shear viscosity by Teaney [20].

- 2001–2003: Work on relativistic equations of motion for viscous fluids by Muronga [21, 22].
- 2004: Magnitude of elliptic flow measured by RHIC experiments, finding overall agreement with relativistic ideal fluid dynamics simulations [23–26].
- 2005: BNL press release: "RHIC Scientists Serve Up 'Perfect' Liquid" [27]: the matter created in relativistic ion collisions behaves like a liquid with a very small viscosity approaching the string theory bound. No constraints on viscosity value.
- 2007: Theory of relativistic viscous fluid dynamics set up as an effective field theory of long-lived, long-wave excitations by two groups [28, 29].
- 2007: 2+1 relativistic viscous fluid modeling of elliptic flow in heavy-ion collisions by several groups [30–33]. First constraints on QCD shear viscosity value, placing it much closer to the strong coupling results than the weak coupling results.
- 2010: The Large Hadron Collider (LHC) at CERN starts operation.
- 2010: Understanding that nuclear geometry fluctuations drive triangular flow in heavy-ion collisions by Alver and Roland [34].
- 2010: Analytic fluid modeling of heavy-ion collisions with two-dimensional flow by Gubser [35].
- 2010: Formulation of relativistic anisotropic hydrodynamics by two groups [36, 37].
- 2010: Fully 3+1d relativistic viscous hydrodynamic simulations of heavy-ion collisions by Schenke, Jeon, and Gale [38].
- 2010: Elliptic flow measured in proton–proton collisions at the LHC [39].
- 2012: Elliptic flow measured in proton-lead collisions at the LHC [40–43].
- 2015: Discovery of an off-equilibrium "hydrodynamic" attractor by Heller and Spalińksi [44].
- 2015: Elliptic flow measured in proton-gold, deuteron-gold, and ^{3}He-gold collisions at RHIC, confirming relativistic viscous hydrodynamics predictions [45].
- 2016: Observation of nonhydrodynamic mode ring-down of a black hole through gravitational wave data by LIGO [46, 47].

1.3 Notation and Conventions

Throughout this work, natural units will be used in which Planck's constant, the speed of light and Boltzmann's constant will be set to unity, $\hbar = c = k_B = 1$.

Much of the theoretical groundwork will be performed in d space-time dimension ($d-1$ spatial dimensions and one time-like dimension). The convention used for the metric tensor is that of the mostly plus convention, and Greek letters $(\mu, \nu, \lambda, \ldots)$ will be used to denote indices of space-time vectors, with $\mu = 0$ denoting time-like directions and the other entries denoting space-like directions. In general, curved space-time will be assumed, but the symbol $g_{\mu\nu}$ may refer to the metric tensor in both curved and flat space-times. The symbol ∇_μ

will indicate a geometric covariant derivative, while a plain coordinate derivative will be denoted as ∂_μ. Gauge-covariant derivatives will be denoted as D_μ.

For space vectors, Latin indices from the middle of the alphabet (i, j, k) will be used, and space-time indices may thus be decomposed as $\mu = (0, i)$. Space vectors also will be denoted by boldface letters, e.g. $\mathbf{v}$ or arrows, e.g. $\vec{v}$ whereas space gradients will be denoted by the symbol ∂ or $\vec{\partial}$.

1.4 Preliminaries

The modern theory of relativistic fluid dynamics relies heavily on the framework of general relativity, even when aiming for exclusive applications to flat Minkowski space-time. Readers unaccustomed with the relation between fluid dynamics and general relativity may appreciate the analogy with electromagnetism, where charges and currents go hand in hand with electric and magnetic fields (cf. Section 1.5). Fluid dynamics concerns itself with the dynamics of energy and momentum, which distort space-time and thus go hand in hand with gravitational fields. Unlike the case of electromagnetism, it is possible and useful to consider fluid dynamics without the associated gravitational fields because gravity is much weaker than the electromagnetic force. However, it is natural to consider fluid dynamics in the presence of gravitational fields which is most easily incorporated through the framework of general relativity.

In order to keep the book self-contained we have therefore added a very brief introduction to the main concepts and equations of relativistic velocities and general relativity in Appendices A and B, respectively. Readers familiar with general relativity may skip this material without detriment, while readers looking for a more detailed discussion on relativity may benefit from consulting one of the excellent textbooks on this topic.

1.5 An Analogy to Fluid Dynamics: Particle Diffusion

As an introduction to many of the concepts discussed in this book, let us consider a simple analogy to fluid dynamics: nonrelativistic diffusion. Specifically, consider a system with a conserved charge (such as the electric charge). Local charge conservation can be written in terms of the charge density n and the associated current density $\vec{j}$ as

$$\partial_t n + \vec{\partial} \cdot \vec{j} = 0\,. \tag{1.1}$$

Because charge is the only conserved quantity, the current density $\vec{j}$ is not a fundamental object. Therefore, $\vec{j}$ must in some way depend on the local charge density n.

In global equilibrium, one can expect the current density to vanish and the charge density to be constant. Deviations from global equilibrium will involve gradients of n and nonvanishing currents. Hence it is natural to consider an expansion of $\vec{j}$ in terms of gradients of n, known as the gradient expansion.[2] The lowest order in this gradient expansion is zeroth order, where $\vec{j}_{(0)} = 0$, because it is not possible to write down a vector in terms of the scalar n. Zeroth-order diffusion therefore corresponds to $\partial_t n = 0$, or static charge density distribution, which in contrast to its fluid dynamics equivalent (the Euler equation) is not very useful in practice.

The next order in the gradient expansion of $\vec{j}$ is first order in gradients. There are two possible structures to first-order gradients of n that one can write down: $\partial_t n$ and $\vec{\partial} n$, of which only the latter is a vector. Hence from the effective field theory expansion, we can expect $\vec{j} \propto \vec{\partial} n$.

However, there is one other structure to first order in gradients contributing to $\vec{j}$ that we could consider if we slightly enlarge our original setup. Because moving electric charges are associated with electromagnetic fields, we should consider gradients of the gauge potentials $\Phi, \vec{A}$ as building blocks in addition to gradients of n. The gauge potentials are typically referred to as "sources," and may be neglected if desired (for instance, if the charge under consideration is not electric and/or the associated gauge fields are very weak). However, in full generality gradients of sources are to be included in the gradient expansion, which to first order in gradients suggests $\vec{j} \propto \vec{E}$ (Ohm's law) where $\vec{E}$ is the "electric" field associated with the gauge potentials.[3] Therefore, to first order in gradients one has

$$\vec{j}_{(1)} = -D\vec{\partial} n + \sigma \vec{E}\,, \tag{1.2}$$

where D, σ are two proportionality coefficients ("transport coefficients") and the sign choices are convention. D, σ are better known as diffusion constant and conductivity, respectively. The effective field theory framework cannot be used to calculate values for D, σ (for this, a particular microscopic theory, such as kinetic theory, must be selected). However, effective field theory *can* be used to obtain relations among transport coefficients, such as the Einstein relations

$$\sigma = D\chi\,, \tag{1.3}$$

[2] Note that no mention of a microscopic description giving rise to $\vec{j}$ is made, such as "currents arise from moving electrons." For this reason, the gradient expansion is universal in the sense that it contains all terms allowed by symmetries alone, which corresponds to the framework of effective field theory. On the other hand, effective field theory cannot be used to derive the quantitative value of the coefficients arising in the gradient expansion (e.g. the dependence of the diffusion constant on the electron charge).

[3] In principle, also $\vec{j} \propto \vec{B}$ is possible, where $\vec{B}$ is the "magnetic" field. However, most systems in nature respect a symmetry known as parity (space direction flips), which forbids $\vec{j} \propto \vec{B}$. We taciturnly assume symmetry under parity in the following, even though the study of parity-violating fluids is a vibrant research subfield in itself.

where χ is the static charge susceptibility.[4] Plugging (1.2) in the equation for charge conservation (1.1), neglecting the source $\vec{E} \to 0$, and assuming D to be constant leads to

$$\partial_t n = D\partial^2 n \,, \tag{1.4}$$

which is the familiar diffusion equation. Based on the derivation above, the diffusion equation (1.4) can be expected to provide a good approximation to the actual evolution of the charge density as long as higher-order gradient corrections are small.

Considering higher-order corrections in the gradient expansion, one encounters terms such as $\vec{j} \propto \vec{\partial}\partial^2 n$, $j \propto \vec{\partial}\left(\vec{\partial}n \cdot \vec{\partial}n\right)$ $j \propto \vec{\partial}\partial^2\partial^2 n$, etc. Since the number of possible combinations of gradients of n increases factorially, it is plausible that the gradient series diverges whenever $\vec{\partial}n \neq 0$. For a divergent series, higher-order corrections are not small for any $\vec{\partial}n \neq 0$, hence requiring small higher-order corrections would imply that the diffusion equation (1.4) is only applicable for static situations where $\partial_t n = 0$.

Given the phenomenal success of using the diffusion equation in a large number of nonstatic situations, clearly the criterion of requiring small higher-order gradient corrections must be too strict. It should be replaced by a different criterion that does justice to the success of the diffusion equation, as well as correctly predicting its breakdown. This book reviews the progress made toward formulating such a criterion for the case of relativistic fluid dynamics.

1.6 Outline of This Book

This books starts by giving a derivation of classical relativistic fluid dynamics from first principles, using only symmetries and the techniques from effective field theory (Section 2.1). In particular, no underlying kinetic theory picture is assumed, which allows studying transport for systems that do not have any well-defined quasi-particles. Familiar results such as the Euler and Navier–Stokes equations and second-order conformal fluid dynamics are obtained in this fashion, and applications such as neutron star structure, cosmology, and Bjorken and Gubser flows are considered (Section 2.2). We then proceed to discuss the divergence of the hydrodynamic gradient expansion, and a nonperturbative definition of hydrodynamics through out-of-equilibrium attractor solutions in Section 2.3. The main criterion for the applicability of fluid dynamics is summarized in form of the Central Lemma of fluid dynamics in this section. Hydrodynamic

[4] As a note to expert readers, a quick way to derive the Einstein relations is to calculate two-point retarded correlation functions $\langle \vec{j}\vec{j}\rangle$, once in the canonical approach (neglecting the sources $\vec{A} = 0$ from the outset) and once in the variational approach where $\langle \vec{j}\vec{j}\rangle \propto \frac{\delta \vec{j}}{\delta \vec{A}}$. The Einstein relations then follow from the fact that both approaches must agree with each other.

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Paul Romatschke , Ulrike Romatschke
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correlation functions and hydrodynamic collective modes are discussed in Section 2.4, showing that the relativistic Navier–Stokes equations are acausal. Section 2.5 describes the standard resummation procedure that cures problems with causality for relativistic viscous fluid dynamics, giving rise to a set of equations of motion that provide the backbone of modern numerical simulations of relativistic viscous fluids. Section 2.6 expands the scope beyond classical fluids by discussing hydrodynamic fluctuations, an action formulation, and the application of quantum field theory techniques to fluid dynamics.

Chapter 3 aims at connecting the effective description of fluid dynamics to particular microscopic theories and/or techniques. For systems with well-defined quasi-particles, Section 3.1 provides kinetic-theory calculations of hydrodynamic transport coefficients and collective modes. The following sections provide a discussion of the corresponding calculations from a strongly coupled systems perspective using the conjectured gauge/gravity duality (Section 3.2), free thermal field theory (Section 3.3), and lattice gauge theory (Section 3.4).

Chapter 4 contains a review of theory components and models that are necessary to apply relativistic fluid dynamics to real-world relativistic nuclear collision systems, including nuclear geometries (Sections 4.3 and 4.4), calculations of energy deposition assuming weak/strong coupling (Sections 4.5 and 4.6), numerical algorithms to solve relativistic viscous fluid dynamics (Section 4.8), hadronization procedures (Section 4.9), and the calculation of experimentally accessible observables (Section 4.10).

Chapter 5 provides theory-data comparisons from relativistic fluid dynamics simulations of nuclear collisions, while Chapter 6 summarizes our understanding of, and prospects for, fluid dynamics theory in the twenty-first century and beyond.